Exercise 10

A Maxima and Minima Problems involving Rectangles, Circles and sectors

1
Solution

(a) The length of $AB = 24$.

$\therefore \quad 2x + 2z = 24$

$z = 12 - x$ .... (1)

The length of $AD = 9$.

$\therefore \quad y + 2z = 9$

$y = 9 - 2z$.......................... (2)

Substitute (1) and (2) into (3)

Volume of the box, $V = xyz$ .... (3)

$= x(2x - 15)(12 - x)$

$= -2x^3 + 39x^2 - 180x$

$\therefore$ volume of the box is $V = -2x^3 + 39x^2 - 180x$. (Shown)

(b) Differentiate V with respect to x

$\frac{\mathrm{d}V}{\mathrm{d}x} = -6x^2 + 78x - 180$ .... (4)

For stationary value of V, $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$

$-6x^2 + 78x - 180 = 0$

$x^2 - 13x + 30 = 0$

$(x - 10)(x - 3) = 0$

$x = 3$ or $x = 10$

When $x = 3$,

$V = -2(3)^3 + 39(3)^2 - 180(3)$

$= -243 < 0$

Hence $x = 10$ is the only answer.

Use Second Derivative test to show V is maximum

Differentiate (4) with respect to x

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -12x + 78$

When $x = 10$,

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -120 + 78 = -42 < 0$

Hence V is maimum when $x = 10$.

2
Solution

The perimeter of the remaining cardboard has a fixed length of L cm.

$\therefore$ Length of the side rectangle cardboard + 2(Length of the height of cardboard) + circumference of semicircle = L

$$2x + 2y + \pi x = L$$

$$(2+\pi)x + 2y = L$$

$$y = \frac{L-(2+\pi)x}{2} \quad \text{........................ (1)}$$

Let A denote the area of the cross section.

$$A = (\text{Area of rectangular cardboard}) - (\text{Area of semi-circle})$$

$$A = 2xy - \frac{\pi x^2}{2} \quad \text{.................................. (2)}$$

Substitute (1) into (2)

$$= 2x\left(\frac{L-(2+\pi)x}{2}\right) - \frac{\pi x^2}{2}$$

$$= Lx - (2+\pi)x^2 - \frac{\pi x^2}{2}$$

$$= Lx - \left(2 + \frac{3\pi}{2}\right)x^2$$

Differentiate A with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = L - 2\left(2 + \frac{3\pi}{2}\right)x$$

For maximum area, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$$L - 2\left(2 + \frac{3\pi}{2}\right)x = 0$$

$$2\left(\frac{4+3\pi}{2}\right)x = L$$

$$x = \frac{L}{4+3\pi} \quad \text{............................... (4)}$$

Substitute (4) into (3)

$$y = \frac{L}{2} - \frac{(2+\pi)}{2}\left(\frac{L}{4+3\pi}\right)$$

$$= \frac{L(2+2\pi)}{2(4+3\pi)}$$

$$= \frac{L(1+\pi)}{4+3\pi} \quad (\text{or } 0.309L)$$

Use Second Derivative test to show A is maximum

Differentiate $\frac{\mathrm{d}A}{\mathrm{d}x}$ with respect to x

$$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = -2\left(2 + \frac{3\pi}{2}\right)$$

Since $\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = -2\left(2 + \frac{3\pi}{2}\right) < 0$, hence A is a maximum.

3
Solution

(a) Let A denotes the total external surface area of the box and the lid

$$A = 3x^2 + 2(3xy) + 2xy + 3x^2 + 2(3kxy) + 2kxy$$
$$= 6x^2 + 8xy + 8kxy$$
$$= 6x^2 + 8xy(1+k) \text{ .... (1)}$$

Let V denotes the volume of the box.

$V = 3x^2 y$

Given that the voulme of the box is $300\ \text{cm}^3$, i.e. $V = 300$

$\therefore \quad 300 = 3x^2 y$

$$y = \frac{100}{x^2} \text{ .... (2)}$$

Substitute (2) into (1)

$$A = 6x^2 + 8x\left[\frac{100}{x^2}(1+k)\right]$$

$$A = 6x^2 + \frac{800}{x}(1+k)$$

Differentiate A with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 12x - \frac{800}{x^2}(1+k)$$

At minimum, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$$12x - \frac{800}{x^2}(1+k) = 0$$

$$x^3 = \frac{200(1+k)}{3}$$

$$x = \left[\frac{200(1+k)}{3}\right]^{\frac{1}{3}}$$

Use Second Derivative test to show A is minimum

Differentiate $\frac{\mathrm{d}A}{\mathrm{d}x}$ with respect to x

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 12 + \frac{1600}{x^3}(1+k)$$

Since $x > 0,\ k > 0$

$$\therefore \frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 12 + \frac{1600}{x^3}(1+k) > 0$$

$x = \left[\frac{200(1+k)}{3}\right]^{\frac{1}{3}}$ gives minimum A

(b) $\dfrac{y}{x} = \dfrac{\dfrac{100}{x^2}}{x}$ $\quad \triangleleft$ from (2): $y = \dfrac{100}{x^2}$

$= \dfrac{100}{x^3}$

$= \dfrac{100}{\dfrac{200(1+k)}{3}}$ $\quad \triangleleft\ x = \left[\dfrac{200(1+k)}{3}\right]^{\frac{1}{3}}$

$= \dfrac{3}{2(1+k)}$ .... (3)

The ratio is $3:2(1+k)$.

(c) Given $\quad 0 < k \le 1$

$1 < 1 + k \le 2$ $\quad \triangleleft$ add 1 on all sides

$2 < 2(1+k) \le 4$ $\quad \triangleleft$ mutliply 2 on all sides

$\dfrac{1}{4} \le \dfrac{1}{2(1+k)} < \dfrac{1}{2}$ $\quad \triangleleft$ reciprocal on all sides

$\dfrac{3}{4} \le \dfrac{3}{2(1+k)} < \dfrac{3}{2}$

$\dfrac{3}{4} \le \dfrac{y}{x} < \dfrac{3}{2}$

$\therefore$ the range of values of $\dfrac{y}{x}$ lies is $\dfrac{3}{4} \le \dfrac{y}{x} < \dfrac{3}{2}$.

(d) For box to have square ends, i.e. $y = x$

From $y = x$, it gives $\dfrac{y}{x} = 1$ .... (4)

Substitute (4) into (3)

$\dfrac{y}{x} = \dfrac{3}{2(1+k)}$

$1 = \dfrac{3}{2(1+k)}$

$1 + k = \dfrac{3}{2}$

$k = 0.5 \quad$ (Shown)

4
Solution

Let V be the volume of the box.

$$V = (\text{horizontal base of the box}) \times (\text{depth of the box})$$
$$= (\pi x^2 + 10x^2)y$$

Given that the volume of the box is 800 cm^3,

$$(\pi x^2 + 10x^2)y = 800$$
$$y = \frac{800}{(\pi+10)x^2} \quad \text{......................} \; (1)$$

Let the external surface area be $A\text{ cm}^2$.

$$A = 10x^2 + 10xy + \pi x^2 + 2\pi xy$$
$$= (\pi+10)x^2 + 2(\pi+5)xy \quad \text{......................} \; (2)$$

Substitute (1) into (2)

$$= (\pi+10)x^2 + 2(\pi+5)x\left[\frac{800}{(\pi+10)x^2}\right]$$
$$= (\pi+10)x^2 + \frac{1600(\pi+5)}{(\pi+10)x} \quad \text{(Shown)} \text{ ....} \; (3)$$

Differentiate (3) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2(\pi+10)x - \frac{1600(\pi+5)}{(\pi+10)x^2}$$

At minimum, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$.

i.e. $2(\pi+10)x - \frac{1600(\pi+5)}{(\pi+10)x^2} = 0$

$$2(\pi+10)x = \frac{1600(\pi+5)}{(\pi+10)x^2}$$
$$x^3 = \frac{800(\pi+5)}{(\pi+10)^2}$$
$$x = \left[\frac{800(\pi+5)}{(\pi+10)^2}\right]^{\frac{1}{3}} = 3.3535$$
$$= 3.35 \text{ (correct to 3 s.f.)}$$

$\therefore$ the value of x is 3.35.

Use Second Derivative test to show A is minimum

Differentiate $\frac{\mathrm{d}A}{\mathrm{d}x}$ with respect to x

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 2(\pi+10)x + \frac{3200(\pi+5)}{(\pi+10)x^3}$$

When $x^3 = \frac{800(\pi+5)}{(\pi+10)^2}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} > 0$

Thus the external surface area is minimum at $x = 3.35$.

5
Solution

(a) Given that time taken to build the fence is 200 hours,

i.e. $2(x+2y)+7\pi\left(\frac{x}{2}\right)=200$

$$y=50-\frac{7\pi x}{8}-\frac{\pi}{2} \text{ .... (1)}$$

Substitute (1) into (2)

Area of flower bed, $A=\frac{1}{2}\pi\left(\frac{x}{2}\right)^2+xy$ .... (2)

$$A=\frac{1}{8}\pi x^2+x\left[50-\frac{7\pi x}{8}-\frac{x}{2}\right]$$

$$A=50x-\frac{3}{4}\pi x^2-\frac{1}{2}x^2 \quad \text{(Shown) .... (3)}$$

(b) Differentiate (3) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x}=50-\left(\frac{3\pi}{2}+1\right)x$$

For maximum A, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$.

i.e. $50-\left(\frac{3\pi}{2}+1\right)x=0$

$$x=\frac{100}{3\pi+2}$$
$$=8.75$$

Substitute $x=8.7529$ into (1).

$$y=50-\left(\frac{7\pi}{8}+\frac{1}{2}\right)(8.7529)$$
$$=21.6$$

The values of x and y are 8.75 and 21.6 respectively.

Use Second Derivative test to show A is maximum

Differentiate $\frac{\mathrm{d}A}{\mathrm{d}x}$ with respect to x

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=-\left(\frac{3\pi}{2}+1\right)$$

$\therefore \frac{\mathrm{d}^2A}{\mathrm{d}x^2}<0$

Hence, A is maximum when $x=8.75$. (Proved)

6
Solution

(a) Area of $\Delta OCD = \frac{1}{2}(OC))(OD)\sin\theta$

$$= \frac{1}{2}(3)(3)\sin\theta$$

$$= \frac{9}{2}\sin\theta$$

Area of sector $OAB = \frac{1}{2}(OA)(OB)\theta$

$$= \frac{1}{2}(1)(1)\theta$$

$$= \frac{1}{2}\theta$$

Area of R = (Area of ΔOCD) − (Area of sector OAB)

$$= \frac{9}{2}\sin\theta - \frac{1}{2}\theta \text{ .... (1)}$$

(b) Differentiate (3) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{9}{2}\cos\theta - \frac{1}{2} \text{ .... (4)}$$

For stationary value of A, $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$.

$$\frac{9}{2}\cos\theta = \frac{1}{2}$$

$$\cos\theta = \frac{1}{9}$$

$$\theta = \cos^{-1}\left(\frac{1}{9}\right)$$

$\theta = 1.46$ radians (correct to 3 SF)

Differentiate (4) with respect to x

$$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = -\frac{9}{2}\sin\theta$$

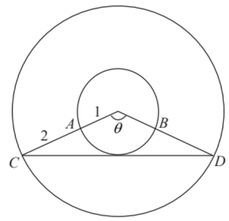
When $\theta = \cos^{-1}\left(\frac{1}{9}\right)$

$$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = -\frac{9}{2}\sin\left(\cos^{-1}\frac{1}{9}\right)$$

$$\therefore \frac{\mathrm{d}^2A}{\mathrm{d}x^2} < 0$$

Hence when $\theta = 1.46$ radians, area of R is greatest.

(c)



θ is largest when chord CD touches the smaller circle.

$$\therefore \quad (OC)\cos\left(\frac{1}{2}\theta\right) = 1$$

$$3\cos\left(\frac{1}{2}\theta\right) = 1$$

$$\cos\left(\frac{1}{2}\theta\right) = \frac{1}{3}$$

$$\frac{1}{2}\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

$$\theta = 2\cos^{-1}\left(\frac{1}{3}\right)$$

$$\theta = 2.46 \text{ radians}$$

$\therefore$ the greatest value of θ is 2.46 radians.

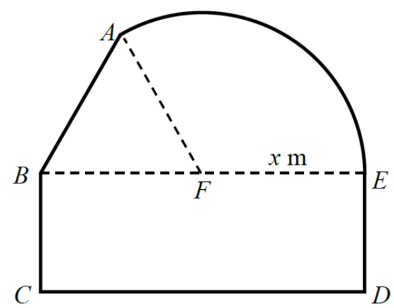
7
Solution

(a) Given ΔBFA is an equilateral triangle, $\angle AFB = 60^\circ$

Angle $AFE = 180^\circ - \angle AFB$
$= 180^\circ - 60^\circ$
$= 120^\circ$

Area of sector $FEA = \frac{120}{360}\pi x^2 = \frac{1}{3}\pi x^2$ m^2

$\therefore$ the area of sector FEA is $\frac{1}{3}\pi x^2$ m^2.



(b) Area of the park = Area of ΔAFB + Area of rectangle $BEDC$ + Area sector AFE

$$= \frac{1}{2}x^2 \sin 60^\circ + BC(2x) + \frac{1}{3}\pi x^2$$

Given that the area of the park is 1000 metres,

$$\therefore \quad 1000 = \frac{1}{2}x^2 \sin 60^\circ + BC(2x) + \frac{1}{3}\pi x^2$$

$$2x(BC) = 1000 - \frac{\sqrt{3}}{4}x^2 - \frac{1}{3}\pi x^2$$

$$BC = \frac{500}{x} - \frac{\sqrt{3}}{8}x - \frac{\pi}{6}x$$

Perimeter of fence use to enclose the park

$P = AB + BC + CD + DE +$ Arc length AE

$$P = x + \left(\frac{500}{x} - \frac{\sqrt{3}}{8}x - \frac{\pi}{6}x\right) + 2x + \left(\frac{500}{x} - \frac{\sqrt{3}}{8}x - \frac{\pi}{6}x\right) + \frac{60}{360}(2\pi x)$$

$$P = \frac{1000}{x} + \frac{x}{12}(4\pi + 36 - 3\sqrt{3}) \text{ (Shown) .... (1)}$$

(c) Differentiate (1) with respect to x

$$\frac{dP}{dx} = \frac{-1000}{x^2} + \frac{1}{12}(36 - 3\sqrt{3} + 4\pi) \text{ .... (2)}$$

At stationary, $\frac{dP}{dx} = 0$.

$$\frac{-1000}{x^2} + \frac{1}{12}(36 - 3\sqrt{3} + 4\pi) = 0$$

$$\frac{1000}{x^2} = \frac{1}{12}(36 - 3\sqrt{3} + 4\pi)$$

$$x^2 = \frac{1000 \times 12}{36 - 3\sqrt{3} + 4\pi}$$

$$x = 16.634$$

Substitute $x = 16.634$ into (1).

$$P = \frac{1000}{16.634} + \frac{16.634}{12}(4\pi + 36 - 3\sqrt{3})$$
$$= 120.2$$

$\therefore$ the stationary value of P is 120.2 m

Use Second Derivative test to show P is minimum

Differentiate (3) with respect to x

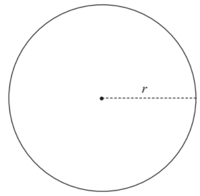
$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{2000}{x^3}$$

When $x = 16.634$, $\dfrac{\mathrm{d}^2P}{\mathrm{d}x^2} = \dfrac{2000}{16.634^3} > 0$

Hence, P is minimum when $x = 16.634$. (Justified)

8
Solution

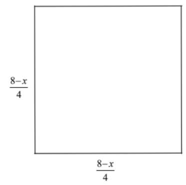
The length x of the wire forms a circle with circumference x cm.
Let r be the radius of the circle.



$\therefore \quad x = 2\pi r$ .... (1)

$$r = \frac{x}{2\pi}$$

The perimeter of the length of the square is $8 - x$.
Let the length of the side of the square be y.



$$\therefore \quad y = \frac{8-x}{4}$$

Sum of areas
A = Area of circle + Area of square

$$A = \pi r^2 + y^2$$

$$A = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{8-x}{4}\right)^2$$

$$A = \frac{x^2}{4\pi} + \frac{1}{16}(8-x)^2$$

Differentiate A with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{2x}{4\pi} + \frac{1}{16}(2)(8-x)(-1)$$

$$= \frac{x}{2\pi} - \frac{1}{8}(8-x) \text{ .... (2)}$$

When A is at minimum, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

i.e.

$$0 = \frac{x}{2\pi} - \frac{1}{8}(8-x)$$

$$0 = \frac{1}{2}\left[\frac{x}{\pi} - \frac{1}{4}(8-x)\right]$$

$$0 = \frac{x}{\pi} - 2 + \frac{x}{4}$$

$$2 = \frac{4x + x}{4\pi}$$

$$x = \frac{8\pi}{4+\pi}$$

Use Second Derivative test to show A is minimum
Differentiate (2) with respect to x

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \frac{1}{2\pi} + \frac{1}{8}$$

$$\therefore \quad \frac{\mathrm{d}^2A}{\mathrm{d}x^2} > 0.$$

Hence A is minimum when $x = \frac{8\pi}{4+\pi}$.

Substitute $x = \dfrac{8\pi}{4+\pi}$ into (1)

$$\frac{8\pi}{4+\pi} = 2\pi r$$

$$r = \frac{4}{4+\pi} \quad \text{(Shown)}$$

Exercise 10

B Maxima and Minima Problems involving Mensuration

9
Solution

Let A be the external surface area of the metal bar.

$A = 300y + 150\pi x + 2xy + 2\pi\left(\frac{x}{2}\right)^2$ .... (1)

Let V be the volume of the metal bar.

$$V = 150xy + 150\pi\left(\frac{x}{2}\right)^2$$

Given that the bar has a volume of 7200 mm^3,

$\therefore 7200 = 150xy + \frac{75}{2}\pi x^2$ ⊲ express y in terms of x

$y = \frac{48}{x} - \frac{\pi x}{4}$ .... (2)

Substitute (2) into (1):

$$A = 300\left(\frac{48}{x} - \frac{\pi x}{4}\right) + 150\pi x + 2x\left(\frac{48}{x} - \frac{\pi x}{4}\right) + \frac{\pi x^2}{2}$$
$$= \frac{14400}{x} + 96 + 75\pi x$$
$$= 14400x^{-1} + 96 + 75\pi x$$

Differentiate A with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = -14400x^{-2} + 75\pi$$

For least surface area, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$$-\frac{14400}{x^2} + 75\pi = 0$$

$$x = \sqrt{\frac{192}{\pi}} \quad \text{or} \quad x = -\sqrt{\frac{192}{\pi}} \quad (\text{rejected} \because x > 0)$$

Differentiate $\frac{\mathrm{d}A}{\mathrm{d}x}$ with respect to x

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 2 \times 14400x^{-3}$$

When $x = \sqrt{\frac{192}{\pi}}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} > 0$

$\therefore$ A is minimum when $x = \sqrt{\frac{192}{\pi}}$.

10
Solution

Let A be the external surface area of the container.

$$A = 2\pi r^2 + 2\pi rh$$

Given that the container's external surface area measures 16200 cm^2,

i.e. $2\pi r^2 + 2\pi rh = 16200$ $\quad \triangleleft$ express h in terms of r

$$h = \frac{8100}{\pi r} - r \text{ .... (1)}$$

Let V be the volume of the container.

$$V = \frac{2}{3}\pi r^3 + \pi r^2 h \text{ .... (2)}$$

Substitute (1) into (2):

$$V = \frac{2}{3}\pi r^3 + \pi r^2 \left(\frac{8100}{\pi r} - r\right)$$

$$= -\frac{1}{3}\pi r^3 + 8100r$$

Differentiate V with respect to x

$$\frac{\mathrm{d}V}{\mathrm{d}r} = -\pi r^2 + 8100 \text{ .... (3)}$$

At maximum V, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$.

$$\therefore \quad -\pi r^2 + 8100 = 0$$

$$\pi r^2 = 8100$$

$$r^2 = \frac{8100}{\pi}$$

$$r = \sqrt{\frac{8100}{\pi}}, \quad \text{since } r > 0$$

Differentiate (3) with respect to x

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -2\pi r$$

When $r = \sqrt{\frac{8100}{\pi}}$, $\frac{\mathrm{d}^2V}{\mathrm{d}r^2} < 0$

Hence, V is maximum when $r = \sqrt{\frac{8100}{\pi}}$.

Substitute $r=\sqrt{\dfrac{8100}{\pi}}$ into (1):

$$\text{Maximum } V=-\frac{1}{3}\pi\frac{8100}{\pi}\sqrt{\frac{8100}{\pi}}+8100\sqrt{\frac{8100}{\pi}}$$

$$=\sqrt{\frac{8100}{\pi}}\left(-\frac{1}{3}\pi\frac{8100}{\pi}+8100\right)$$

$$=\frac{90}{\sqrt{\pi}}(5400)$$

$$=\frac{486000}{\sqrt{\pi}}$$

The maximum volume of this container is $\dfrac{486000}{\sqrt{\pi}}$ cm^3.

11
Solution

Volume of the remaining solid, V = Volume of the cuboid − Volume of the cylinder

$$= (2x)^2 y - \pi\left(\frac{x}{2}\right)^2 y$$

Given that V is $\frac{1}{4}$ m^3,

$$\therefore \quad (2x)^2 y - \pi\left(\frac{x}{2}\right)^2 y = \frac{1}{4} \quad \triangleleft \text{ express } y \text{ in terms of } x$$

$$x^2 y\left(4 - \frac{\pi}{4}\right) = \frac{1}{4}$$

$$y = \left(\frac{1}{16-\pi}\right)\frac{1}{x^2} \quad \text{.........................} \ (1)$$

Total surface area, A

$$= 4(2xy) + 2\left[(2x)^2 - \pi\left(\frac{\pi}{2}\right)^2\right] + 2\pi\left(\frac{\pi}{2}\right)y$$

$$= \left(8 - \frac{\pi}{2}\right)x^2 + (8+\pi)xy \quad \text{..............................} \ (2)$$

Substitute (1) into (2):

$$= \left(8 - \frac{\pi}{2}\right)x^2 + (8+\pi)x\left(\frac{1}{16-\pi}\right)\frac{1}{x^2}$$

$$= \left(8 - \frac{\pi}{2}\right)x^2 + \left(\frac{8+\pi}{16-\pi}\right)\frac{1}{x} \quad \text{(Shown)}$$

$$= \left(8 - \frac{\pi}{2}\right)x^2 + \left(\frac{8+\pi}{16-\pi}\right)x^{-1} \text{..............................} \ (3)$$

Differentiate (3) with respect to x

$$\frac{dA}{dx} = 2\left(8 - \frac{\pi}{2}\right)x - \left(\frac{8+\pi}{16-\pi}\right)x^{-2}$$

$$= (16-\pi)x - \left(\frac{8+\pi}{16-\pi}\right)\frac{1}{x^2} \quad \text{.....................} \ (4)$$

For A to be maximum or minimum, $\frac{dA}{dx} = 0$.

$$\therefore \quad (16-\pi)x - \left(\frac{8+\pi}{16-\pi}\right)\frac{1}{x^2} = 0$$

$$\frac{(16-\pi)^2 x^3 - (8+\pi)}{(16-\pi)x^2} = 0$$

$$x^2 = \frac{8+\pi}{(16-\pi)^2}$$

Differentiate (4) with respect to x

$$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = 16 - \pi + 2\left(\frac{8+\pi}{16-\pi}\right)\frac{1}{x^3}$$

$$= 16 - \pi + 2\left(\frac{8+\pi}{16-\pi}\right)\frac{(16-\pi)^2}{8+\pi}$$

$$= 16 - \pi + 2(16-\pi)$$

$$= 48 - 3\pi$$

$$= 38.6$$

$$\therefore \ \frac{\mathrm{d}^2 A}{\mathrm{d}x^2} > 0$$

Therefore A is minimum when $x^3 = \dfrac{8+\pi}{(16-\pi)^2}$. (Shown)

12
Solution

(a) Let the total surface area of metal sheet to make the can be A.

$A =$ curved surface area of the can + base area of the can

$= 2\pi xy + 2\pi x^2$

Given that the total area of metal sheet used to make the can is $40\pi^2\ \text{cm}^2$,

i.e. $2\pi xy + 2\pi x^2 = 40\pi^2$ ◁ express y in terms of x

$$xy + x^2 = 20\pi$$

$$xy = 20\pi - x^2$$

$$y = \frac{20\pi - x^2}{x}$$

$$= \frac{20\pi}{x} - x \ \text{......................... (1)}$$

Volume of the can, V

$=$ Volume of the cylinder $-$ Volume of the hemisphere

$$= \pi x^2 y - \frac{4}{3}\left(\frac{1}{2}\pi x^3\right) \ \text{........................... (2)}$$

Substitute (1) into (2):

$$= \pi x^2\left(\frac{20\pi}{x} - x\right) - \frac{2}{3}\pi x^3$$

$$= \pi x(20\pi - x^2) - \frac{2}{3}\pi x^3$$

$$= \pi x\left[20\pi - x^2 - \frac{2}{3}x^2\right]$$

$$= \pi x\left[20\pi - \frac{5}{3}x^2\right] \quad \text{(Shown) .... (3)}$$

(b) Differentiate (3) with respect to x

$$\frac{dV}{dx} = \pi\left[20\pi - \frac{5}{3}x^2\right] + \pi x\left[-\frac{10}{3}x\right]$$

At maximum volume, $\frac{dV}{dx} = 0$

$$\therefore \quad \pi\left[20\pi - \frac{5}{3}x^2\right] + \pi x\left[-\frac{10}{3}x\right] = 0$$

$$20\pi - \frac{5}{3}x^2 - \frac{10}{3}x^2 = 0$$

$$20\pi - 5x^2 = 0$$

$$5x^2 = 20\pi$$

$$x^2 = 4\pi$$

$$x = \sqrt{4\pi}\ \text{cm or} -\sqrt{4\pi}\ \text{cm (Rejected since } x > 0)$$

Use first derivative test to check if V is maximum:

x	3.535	$\sqrt{4\pi} \approx 3.54$	3.545
$\frac{dV}{dx}$	$1.1018 > 0$	0	$-0.0103 < 0$

Therefore the volume is maximum at $x = \sqrt{4\pi}$.

Alternative Method

Use second derivative test to check if V is maximum:

Differentiate $\frac{dV}{dx}$ with respect to x

$$\frac{d^2V}{dx^2} = \pi\left[-\frac{10}{3}x\right] - \frac{20\pi}{3}x$$

When $x = \sqrt{4\pi}$,

$$\frac{d^2V}{dx^2} = -111.37$$

$$\therefore \frac{d^2V}{dx^2} < 0$$

Therefore the volume is maximum at $x = \sqrt{4\pi}$.

Substitute $x = \sqrt{4\pi}$ into (1):

$$V = \pi\sqrt{4\pi}\left(20\pi - \frac{5}{3}(4\pi)\right)$$

$$= 466.49 \text{ cm}^3$$

$\therefore$ the maximum volume is 466.49 cm^3.

(c) We assume negligible thickness for the metal sheet used.

13
Solution

(a) Let the height of the cylinder be h cm. Hence height of the cone is $(H-h)$ cm.

Area of the curved surface area of the cylinder + area of the disc $= 2\pi rh + \pi r^2$

Given that the are of the cover and the curved surface of the cylinder is $H^2\pi$,

$\therefore \quad \pi r^2 + 2\pi rh = H^2\pi \quad \triangleleft$ express h in terms of r and H

$$h = \frac{H^2 - r^2}{2r}$$
$$= \frac{H^2}{2r} - \frac{r}{2} \text{ .... (1)}$$

Volume of the tank, V = volume of cylinder + volume of cone

$$= \pi r^2 h + \frac{1}{3}\pi r^2 (H-h)$$
$$= \frac{2}{3}\pi r^2 h + \frac{1}{3}\pi r^2 H \text{ .... (2)}$$

Substitute (1) into (2):

$$= \frac{2}{3}\pi r^2 \left(\frac{H^2}{2r} - \frac{r}{2}\right) + \frac{1}{3}\pi r^2 H$$
$$V = \frac{\pi}{3}(H^2 r - r^3 + Hr^2) \quad \text{(Shown) .... (3)}$$

(b) Differentiate (3) with respect to x

$$\frac{dV}{dr} = \frac{\pi}{3}(H^2 + 2rH - 3r^2)$$

At stationary value of V, $\frac{dV}{dr} = 0$.

$$H^2 + 2rH - 3r^2 = 0$$
$$r = \frac{-2H \pm \sqrt{4H^2 + 12H^2}}{-6}$$
$$= \frac{1}{3}(H \pm 2H)$$

Since $H > 0, r > 0$,

$$r = \frac{1}{3}(H + 2H)$$
$$= H$$

Use second derivative test to check if V is maximum:

Differentiate $\frac{dV}{dx}$ with respect to x

$$\frac{d^2V}{dr^2} = \frac{\pi}{3}(2H - 6r)$$

For $r = H$,

$$\frac{d^2V}{dr^2} = \frac{\pi}{3}(2H - 6H)$$
$$= -\frac{4H\pi}{3} < 0 \text{ as } H > 0.$$

Therefore, V gives a maximum value when $r = H$.

14
Solution

(a) Volume of the stylus = volume of cylinder + volume of hemisphere

$$= \pi r^2 s + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) \text{ .... (*)}$$

Gven that the volume of the stylus is fixed at k,

$$\therefore \quad k = \pi r^2 s + \frac{2}{3}\pi r^3$$

$$s = \frac{k}{\pi r^2} - \frac{2r}{3} \text{ .... (1)}$$

Let A be the total external surface area of the stylus

A = curved surface area of the cylinder + area of the circular disc + curved surface area of the hemisphere

$$= 2\pi r s + \pi r^2 + \frac{1}{2}\left(4\pi r^2\right) \text{......................... (**)}$$

Substitute (1) into (2):

$$= 2\pi r\left(\frac{k}{\pi r^2} - \frac{2r}{3}\right) + 3\pi r^2$$

$$= \frac{2k}{r} - \frac{4\pi r^2}{3} + 3\pi r^2$$

$$= \frac{2k}{r} + \frac{5}{3}\pi r^2 \text{ .... (3)}$$

Differentiate (3) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{10}{3}\pi r - \frac{2k}{r^2}$$

At minimum value, $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$

$$\therefore \quad \frac{10}{3}\pi r - \frac{2k}{r^2} = 0$$

$$\frac{10}{3}\pi r = \frac{2k}{r^2}$$

$$\frac{1}{3}\pi r^3 = \frac{1}{5}k$$

$$r = \sqrt[3]{\frac{3k}{5\pi}}$$

Substitute $r = \sqrt[3]{\frac{3k}{5\pi}}$ into $k = \pi r^2 s + \frac{2}{3}\pi r^3$:

$\therefore \qquad k = \pi\left(\sqrt[3]{\dfrac{3k}{5\pi}}\right)^2 s + \dfrac{2}{3}\pi\left(\dfrac{3k}{5\pi}\right)$

$$k = \pi\left(\sqrt[3]{\frac{3k}{5\pi}}\right)^2 s + \frac{2}{5}k$$

$$\pi\left(\sqrt[3]{\frac{3k}{5\pi}}\right)^2 s = \frac{3k}{5}$$

$$s = \frac{3k}{5\pi}\left(\frac{5\pi}{3k}\right)^{\frac{2}{3}}$$

$$= \sqrt[3]{\frac{3k}{5\pi}}$$

The value of r is $\sqrt[3]{\dfrac{3k}{5\pi}}$ and the value of s is $\sqrt[3]{\dfrac{3k}{5\pi}}$

(b) Given that the volume of the model is 1270 mm^3, i.e. $k = 1270$.

Substitute $k = 1270$ into (*):

$$1270 = \pi r^2 s + \frac{2}{3}\pi r^3$$

Given that the external surface area of the model is 1290 mm^2, i.e. $A = 1290$

Substitute $A = 1290$ into (**):

$$1290 = 2\pi rs + \pi r^2 + \frac{1}{2}\left(4\pi r^2\right)$$

$$s = \frac{1290 - 3\pi r^2}{2\pi r}$$

Substitute (5) into (4)

$$1270 = \pi r^2\left(\frac{1290 - 3\pi r^2}{2\pi r}\right) + \frac{2}{3}\pi r^3$$

$$\frac{5}{6}\pi r^2 - 645r + 1270 = 0$$

Using GC,

$r = 2.0015, \quad 14.599 \quad \text{or} \quad -16.601$

Since $r > 0$, $r = -16.6$ is rejected

If $r = 2.0015$,

$$s = \frac{k}{\pi(2.0015)^2} - \frac{2(2.0015)}{3} = 99.573$$

If $r = 14.599$, $s = -7.8364$

Hence $r = 14.599$ is rejected since s cannot be negative.

So the only value of r is 2.00.

15
Solution

Let r, h and S be the radius, the height and the surface area of the cylinder respectively.

Volume closed cylinder = $\pi r^2 h$

Given that the volume of the closed cylinder is p,

i.e. $p = \pi r^2 h$

$$h = \frac{p}{\pi r^2} \text{ .... (1)}$$

$$S = 2\pi r^2 + 2\pi rh \text{ .... (2)}$$

Substitute (1) into (2):

$$= 2\pi r^2 + 2\pi r\left(\frac{p}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2p}{r} \text{ .... (3)}$$

Differentiate (3) with respect to x

$$\frac{dS}{dr} = 4\pi r - \frac{2p}{r^2}$$

For mininum of S, $\frac{dS}{dr} = 0$

i.e. $4\pi r - \frac{2p}{r^2} = 0$

$$r = \left(\frac{p}{2\pi}\right)^{\frac{1}{3}}$$

The value of r is $\left(\frac{p}{2\pi}\right)^{\frac{1}{3}}$ cm.

Use second derivative test to check if V is maximum:

Differentiate $\frac{dV}{dx}$ with respect to x

$\frac{d^2S}{dr^2} = 4\pi + \frac{4p}{r^3} > 0$ since r and p are positive.

Therefore, S gives a maximum value when $r = \left(\frac{p}{2\pi}\right)^{\frac{1}{3}}$.

Exercise 10

C Maxima and Minima Problems involving Pythagoras theorem

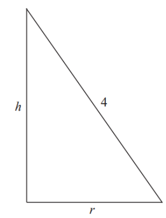
16
Solution

(a) $V =$ Volume of cone $+$ Volume of hemisphere

$$V = \frac{1}{3}\pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3 \text{.....................................} \ (1)$$

Use Pythagoras Theorem,



$$h^2 + r^2 = 4^2$$

$$h = \sqrt{16 - r^2} \text{ ....} \ (2)$$

Substitute (2) into (1)

$$V = \frac{1}{3}\pi r^2 \sqrt{16 - r^2} + \frac{2}{3}\pi r^3$$

Differentiate V with respect to r

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{1}{3}\pi\left(2r\sqrt{16 - r^2} + r^2\left(\frac{1}{2}(16 - r^2)^{-\frac{1}{2}}(-2r)\right)\right) + \frac{2}{3}(3\pi r^2)$$

$$= \frac{1}{3}\pi\left(2r\sqrt{16 - r^2} - \frac{r^3}{\sqrt{16 - r^2}}\right) + 2\pi r^2 \text{ ....} \ (3)$$

For maximum V, $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 0$,

$$\therefore \ \frac{1}{3}\pi\left(2r\sqrt{16 - r^2} - \frac{r^3}{\sqrt{16 - r^2}}\right) + 2\pi r^2 = 0$$

$$2r\sqrt{16 - r^2} - \frac{r^3}{\sqrt{16 - r^2}} + 6r^2 = 0$$

$$2r(16 - r^2) + r^3 + 6r^2\sqrt{16 - r^2} = 0$$

$$2(16 - r^2) - r^2 + 6r\sqrt{16 - r^2} = 0, \ r \neq 0$$

$$32 - 2r^2 - r^2 = -6r\sqrt{16 - r^2}$$

$$32 - 3r^2 = -6r\sqrt{16 - r^2}$$

$$(32 - 3r^2)^2 = (-6r\sqrt{16 - r^2})^2$$

$$1024 - 192r^2 + 9r^4 = 036r^2(16 - r^2)$$

$$1024 - 192r^2 + 9r^4 = 0576r^2 - 36r^4$$

$$\therefore \ 45r^4 - 768r^2 + 1024 = 0 \quad \text{(Shown)}$$

(b) Using GC to solve $45r^4 - 768r^2 + 1024 = 0$

$$r = 3.9508 \approx 3.951 \text{ cm} \quad (3 \text{ d.p})$$

$$r = -3.9508 \approx -3.951 \quad (\text{Rejected, since } r > 0)$$

$$r = 1.2074 \approx 1.207 \text{ cm} \quad (3. \text{ d.p})$$

$$r = -1.2074 \approx -1.207 \quad (\text{Rejected, since } r > 0)$$

$\therefore$ the two solutions are $r = 3.951$ cm or 1.207 cm. (3 d.p)

(c) For $r = 3.9508$, substitute $r = 3.9508$ into (3)

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{1}{3}\pi\left(2(3.9508)\sqrt{16-3.9508^2} - \frac{3.9508^3}{\sqrt{16-3.9508^2}}\right) + 2\pi(3.9508)^2$$

$$= -0.003 \approx 0$$

For $r = 1.2074$, substitute $r = 1.2074$ into (3)

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{1}{3}\pi\left(2(1.2074)\sqrt{16-1.2074^2} - \frac{1.2074^3}{\sqrt{16-1.2074^2}}\right) + 2\pi(1.2074)^2$$

$$= 18.320 \neq 0$$

$\therefore$ there is only $r = 3.951$ cm gives a stationary value of V

$\therefore$ $r_1 = 3.951$ cm

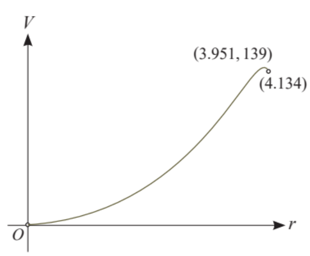
Substitute $r_1 = 3.951$ into (2)

$$h = \sqrt{16 - 3.9508^2}$$

$$= 0.62544$$

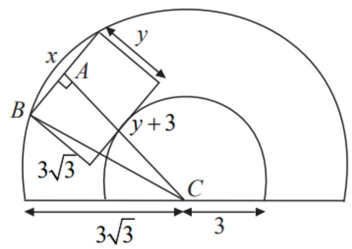
$$\approx 0.625 \text{ cm} \quad (3 \text{ s.f})$$

(d)



17
Solution

(a) By Pythagoras Theorem,



$$AC^2 + AB^2 = BC^2$$
$$(3+y)^2 + x^2 = (3\sqrt{3})^2$$
$$9 + 6y + y^2 + x^2 = 27$$
$$x^2 = 18 - 6y - y^2$$

Area of the sheet, $A = 2xy$

$$= 2y\sqrt{18 - 6y - y^2}$$
$$= 2\sqrt{18y^2 - 6y^3 - y^4} \text{ .... (1)}$$

(b) From (1): $A = 2\sqrt{18y^2 - 6y^3 - y^4}$

Square both sides,

$$A^2 = 4(18y^2 - 6y^3 - y^4)$$
$$2A\frac{\mathrm{d}A}{\mathrm{d}y} = 4(36y - 18y^2 - 4y^3)$$
$$A\frac{\mathrm{d}A}{\mathrm{d}y} = 4y(18 - 9y - 2y^2)$$

For maximum A, $\frac{\mathrm{d}A}{\mathrm{d}y} = 0$

$$4y(18 - 9y - 2y^2) = 0$$
$$2y^2 + 9y - 18 = 0 \quad \text{or} \quad y = 0 \text{ (rejected as } y > 0)$$
$$(2y-3)(y+6) = 0$$
$$y = \frac{3}{2} \text{ or } y = -6 \text{ (rejected as } y > 0)$$

Use First Derivative test to show maximum

y	$\left(\frac{3}{2}\right)^-$	$\left(\frac{3}{2}\right)$	$\left(\frac{3}{2}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}y}$	>0	0	<0

$\therefore$ A is maximum when $y = \frac{3}{2}$. (Proved)

When $y = \frac{3}{2}$, substitute $y = \frac{3}{2}$ into (1)

$$A = 2\sqrt{18\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^4}$$
$$= 7.79$$

$\therefore$ the maximum value of A is 7.79 m^2

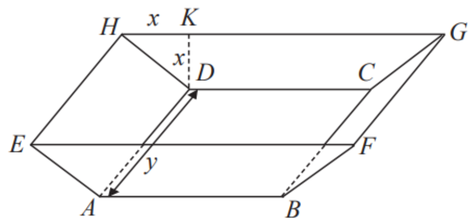
18
Solution

(a) Given that the capacity of the tray is 1980 cm^3,

$$1980 = (\text{area of trapezium}) \times (\text{length})$$
$$= \frac{1}{2}(10x + 12x)y$$
$$y = \frac{1980}{11x^2}$$
$$= \frac{180}{x^2}$$

By Pythagoras Theorem

$$HD^2 = DK^2 + HK^2$$
$$HD^2 = x^2 + x^2$$
$$HD = x\sqrt{2}$$



Total surface area of the tray

$$A = 2(\text{Area of trapezium}) + 2(\text{Area of rectangle}) + \text{area of rectangular base}$$
$$= 22x^2 + 2xy\sqrt{2} + 10xy$$
$$= 22x^2 + 2xy(5 + \sqrt{2})$$
$$= 22x^2 + \frac{360}{x}(5 + \sqrt{2}) \quad \text{(Shown)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

(b) Differentiate (1) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 44x - \frac{360(5 + \sqrt{2})}{x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

At stationary point, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$.

$$\therefore\ 44x - \frac{360(5 + \sqrt{2})}{x^2} = 0$$

Using GC, $x = 3.7440$

Use Second Derivative Test to show minimum

Differentiate (2) with respect to x

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 44 + \frac{720(5 + \sqrt{2})}{x^3}$$

For all $x > 0$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 44 + \frac{720(5 + \sqrt{2})}{x^3} > 0$

$\therefore$ A is a minimum.

Alternative Method

Use Second derivative test to show minimum

$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 44 + \frac{720(5 + \sqrt{2})}{x^3} > 0$ for all $x > 0$.

$\therefore$ A is a minimum.

Minimum A is 925 cm^2 when $x = 3.7440$

19
Solution

(a) From diagram 1, arc length = $a\theta$
From diagram 2, circumference of circle = $2\pi r$

Observe that the are length of the sector in Diagram 1 is also the circumference of the circle in Diagram 2.

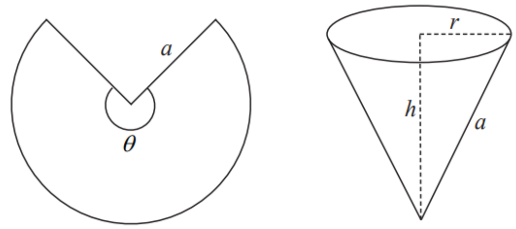


Diagram 1 Diagram 2

Thus $a\theta = 2\pi r$

$$r = \frac{a\theta}{2\pi} \text{ .... (1)}$$

By Pythagoras Theorem

$$a^2 = h^2 + r^2 \text{ .... (2)}$$

Substitute (1) into (2)

$$a^2 = h^2 + \left(\frac{a\theta}{2\pi}\right)^2$$

$$h = \sqrt{a^2 - \left(\frac{a\theta}{2\pi}\right)^2} \text{ .... (3)}$$

(b) $V = \frac{1}{3} \times \text{base area} \times \text{height}$

$$= \frac{1}{3} \times (\pi r^2) \times h \text{ .... (4)}$$

Substitute (1) and (3) into (4)

$$= \frac{1}{3} \times \left[\pi \left(\frac{a\theta}{2\pi}\right)^2\right] \times \sqrt{a^2 - \left(\frac{a\theta}{2\pi}\right)^2}$$

$$= \frac{1}{3} \times \frac{a^2\theta^2}{4\pi^2} \times \sqrt{a^2 - \frac{a^2\theta^2}{4\pi^2}}$$

$$= \frac{a^2\theta^2}{12}\sqrt{\frac{a^2}{4\pi^2}(4\pi^2 - \theta^2)}$$

$$= \frac{a^2\theta^2}{12} \times \frac{a}{2\pi}\sqrt{(4\pi^2 - \theta^2)}$$

$$= \frac{a^3\theta^2}{24\pi^2}\sqrt{(4\pi^2 - \theta^2)} \quad \triangleleft \text{ square both sides}$$

$$V^2 = \frac{a^6\theta^4}{576\pi^2}(4\pi^2 - \theta^2) \quad \text{(Shown) .... (5)}$$

(c) Given that radius is 2, substitute $a = 2$ into (5)

$$V^2 = \frac{2^6\theta^4}{576\pi^2}(4\pi^2 - \theta^2)$$

$$V^2 = \frac{1}{9\pi^4}(4\pi^2\theta^4 - \theta^6)$$

Differentiate V with respect to θ

$$2V\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{1}{9\pi^4}(16\pi^2\theta^3 - 6\theta^5)$$

$$= \frac{2\theta^3}{9\pi^4}(8\pi^2 - 3\theta^2)$$

At maximum volume of the container, $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0.$

i.e. $\frac{2\theta^3}{9\pi^4}(8\pi^2 - 3\theta^2) = 0$

$$\theta^3(8\pi^2 - 3\theta^2) = 0$$

$\theta = 0$ (Rejected since $\theta > 0$) or $\theta = \sqrt{\frac{8\pi^2}{3}}$ or $\theta = -\sqrt{\frac{8\pi^2}{3}}$ (Rejected since $\theta > 0$)

Substitute $\theta = \sqrt{\frac{8\pi^2}{3}}$ into (5)

$$\therefore \quad V = \frac{8\pi^2}{3}\left(\frac{1}{3\pi^2}\right)\sqrt{\left(4\pi^2 - \frac{8\pi^2}{3}\right)}$$

$$= \frac{8}{9}\sqrt{\frac{4\pi^2}{3}}$$

$$= \frac{16\sqrt{3}}{27}\pi$$

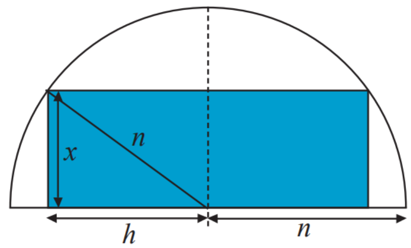
20
Solution

(a) Let h be half the height of the cylinder.

By Pythagoras Theorem

$$h^2 + x^2 = n^2$$
$$h^2 = n^2 - x^2$$
$$h = \sqrt{n^2 - x^2}$$

$\therefore$ the height of the cylinder $= 2\sqrt{n^2 - x^2}$



Let A be the surface area of the cylinder

$$A = 2\pi(\text{radius})(\text{height of the cylinder})$$
$$A = \pi(\text{diameter})(\text{height of the cylinder})$$
$$= \pi x(2\sqrt{n^2 - x^2})$$
$$= 2\pi x\sqrt{n^2 - x^2}$$
$$A = 2\pi x\sqrt{n^2 - x^2} \quad \text{(Shown)} \text{ .... (1)}$$

(b) Differentiate (1) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2\pi\sqrt{n^2 - x^2} + \frac{(2\pi x)\frac{1}{2}(-2x)}{\sqrt{n^2 - x^2}}$$
$$= 2\pi\sqrt{n^2 - x^2} - \frac{2\pi x}{\sqrt{n^2 - x^2}}$$
$$= \frac{2\pi(n^2 - x^2) - 2\pi x^2}{\sqrt{n^2 - x^2}}$$
$$= \frac{2\pi n^2 - 4\pi x^2}{\sqrt{n^2 - x^2}}$$

For maximum surface area of the cylinder, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$$\frac{2\pi n^2 - 4\pi x^2}{\sqrt{n^2 - x^2}} = 0$$
$$2\pi n^2 - 4\pi x^2 = 0$$
$$n^2 - 2x^2 = 0$$
$$x^2 = \frac{n^2}{2}$$
$$x = \frac{n}{\sqrt{2}} \quad \text{or} \quad x = -\frac{n}{\sqrt{2}} \text{ (Rejected since } n > 0)$$

Use Second Derivative Test to show maximum

$$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = \frac{\sqrt{n^2 - x^2}(-8\pi x) - (2\pi n^2 - 4\pi x^2)\dfrac{1}{2\sqrt{n^2 - x^2}}(-2x)}{(n^2 - x^2)}$$

$$= \frac{(n^2 - x^2)(-8\pi x) + x(2\pi n^2 - 4\pi x^2)}{(n^2 - x^2)\sqrt{n^2 - x^2}}$$

$$= \frac{2\pi x(2x^2 - 3n^2)}{(n^2 - x^2)\sqrt{n^2 - x^2}}$$

At $x = \dfrac{n}{\sqrt{2}}$,

$$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = \frac{2\pi\left(\dfrac{n}{2}\right)(n^2 - 3n^2)}{\left(n^2 - \dfrac{n^2}{2}\right)\sqrt{n^2 - \dfrac{n^2}{2}}} < 0$$

Hence A is a maximum at $x = \dfrac{n}{\sqrt{2}}$.

(**Alternative Method**) Use First Derivative Test to show maximum

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{2\pi(n - \sqrt{2}x)(n + \sqrt{2}x)}{\sqrt{n^2 - x^2}}$$

For any $x > 0$, $(n + \sqrt{2}x) > 0$ and $\sqrt{n^2 - x^2} > 0$

When $x < \dfrac{n}{\sqrt{2}}$, $(n - \sqrt{2}x) < 0$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{2\pi(n - \sqrt{2}x)(n + \sqrt{2}x)}{\sqrt{n^2 - x^2}} < 0$$

When $x > \dfrac{n}{\sqrt{2}}$, $(n - \sqrt{2}x) > 0$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{2\pi(n - \sqrt{2}x)(n + \sqrt{2}x)}{\sqrt{n^2 - x^2}} > 0$$

x	$\left(\dfrac{n}{\sqrt{2}}\right)^-$	$\left(\dfrac{n}{\sqrt{2}}\right)$	$\left(\dfrac{n}{\sqrt{2}}\right)^+$
Sign of $\dfrac{\mathrm{d}A}{\mathrm{d}x}$	< 0 $(n - \sqrt{2}x) < 0$	0	> 0

Hence A is a maximum at $x = \dfrac{n}{\sqrt{2}}$.

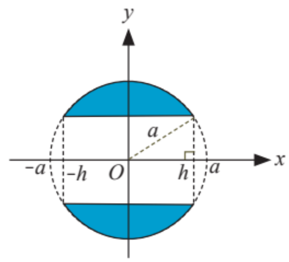
(c) $\dfrac{\text{diameter}}{\text{height}} = \dfrac{\dfrac{n}{2}}{2\sqrt{n^2 - \dfrac{n^2}{2}}}$

$$= \dfrac{\dfrac{n}{\sqrt{2}}}{2\sqrt{\dfrac{n^2}{2}}}$$

$$= \dfrac{1}{2}$$

$\therefore$ the ratio of diameter to height is $1:2$

21
Solution



(a) Let r be the radius of cylinder

By Pythagoras Theorem

$r^2 + h^2 = a^2$

$r = \sqrt{a^2 - h^2}$ .... (1)

S = Internal cylindrical area + external curved surface area of the spherical ring (given)

$= 2\pi rh + 4\pi ah$ .... (2)

Substitute (1) into (2)

$\therefore \quad S = 4\pi ah + 4\pi h\sqrt{a^2 - h^2}$

$= 4\pi h(a + \sqrt{a^2 - h^2})$ (Shown) .... (3)

(b) Differentiate (3) with respect to h

$$\frac{\mathrm{d}S}{\mathrm{d}h} = 4\pi(a + \sqrt{a^2 - h^2}) + 4\pi h \times \frac{1}{2}\left(\frac{-2h}{\sqrt{a^2 - h^2}}\right)$$

$$= 4\pi(a + \sqrt{a^2 - h^2}) - \frac{4\pi h^2}{\sqrt{a^2 - h^2}}$$

$$= 4\pi\left[(a + \sqrt{a^2 - h^2}) - \frac{4\pi h^2}{\sqrt{a^2 - h^2}}\right]$$

$$= 4\pi\left(a + \frac{a^2 - h^2 - h^2}{\sqrt{a^2 - h^2}}\right)$$

$$= 4\pi\left(a + \frac{a^2 - 2h^2}{\sqrt{a^2 - h^2}}\right)$$

For maximum value of S, let $\frac{\mathrm{d}S}{\mathrm{d}h} = 0$

$$4\pi\left(a + \frac{a^2 - 2h^2}{\sqrt{a^2 - h^2}}\right) = 0$$

$$\frac{a^2 - 2h^2}{\sqrt{a^2 - h^2}} = -a$$

$$\frac{2h^2 - a^2}{\sqrt{a^2 - h^2}} = a$$

$$(2h^2 - a^2)^2 = a^2(a^2 - h^2)$$

$$4h^4 - 4h^4a^2 + a^4 = a^4 - a^2h^2$$

$$4h^4 - 3h^4a^2 = 0$$

$$h^2(4h^4 - 3a^2) = 0$$

$$h = \frac{\sqrt{3}}{2}a \quad \text{or} \quad h = -\frac{\sqrt{3}}{2} \text{ (Rejected since } h > 0) \quad h = 0 \text{ (Rejected since } h > 0)$$

Substitute $h = \frac{\sqrt{3}}{2}a$ into (3)

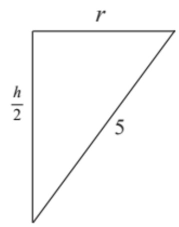
$$\text{Max value of } S = 4\pi\left(\frac{\sqrt{3}}{2}a\right)\left(a + \sqrt{a^2 - \frac{3}{4}a^2}\right)$$

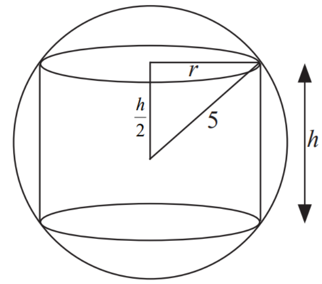
$$= 2\sqrt{3}\pi a\left(a + \frac{1}{2}a\right)$$

$$= 3\sqrt{3}\pi a^2$$

22
Solution

(a) By Pythagoras Theorem





$$r^2 + \left(\frac{h}{2}\right)^2 = 5^2$$

$$r^2 + \frac{h^2}{4} = 25$$

$$4r^2 + h^2 = 100$$

$\therefore$ since $h > 0$, $h = \sqrt{100 - 4r^2}$

$$= \sqrt{4(25 - r^2)}$$

$$= 2\sqrt{25 - r^2} \text{ .... (1)}$$

$V = \pi r^2 h$ .... (2)

Substitute (1) into (2)

$$= \pi r^2 (2\sqrt{25 - r^2})$$

$$= 2\pi r^2 (\sqrt{25 - r^2}) \quad \text{(Shown) .... (3)}$$

(b) Differentiate (3) with respect to r

$$\frac{dV}{dr} = 2\pi \left[2r\sqrt{25 - r^2} + r^2 \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{25 - r^2}}\right)(-2r) \right]$$

For maximum volume, $\frac{dV}{dr} = 0$

$$2r\sqrt{25 - r^2} - \frac{r^3}{\sqrt{25 - r^2}} = 0$$

$$2r\sqrt{25 - r^2} = \frac{r^3}{\sqrt{25 - r^2}}$$

$$2r\sqrt{25 - r^2} = r^3$$

$$50r - 3r^3 = 0$$

$$r^2 = \frac{50}{3}, \ r \neq 0$$

since $r > 0$, $r = \sqrt{\frac{50}{3}} = \frac{5\sqrt{6}}{3}$

Use First Derivative Test to show maximum

r	$\sqrt{\frac{50}{3}}^{-}$	$\sqrt{\frac{50}{3}}$	$\sqrt{\frac{50}{3}}^{+}$
$\frac{dV}{dr}$	/	—	\

$\therefore$ $r = \sqrt{\frac{50}{3}}$ gives a maximum volume.

(**Alternative Method**) Use Second Derivative Test to show maximum

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -217.65 < 0$$

$\therefore\ r = \sqrt{\frac{50}{3}}$ gives a maximum volume.

Substitute $r = \sqrt{\frac{50}{3}}$ into (3)

$$\text{Maximum volume, } V = 2\pi\left(\sqrt{\frac{50}{3}}\right)^2\sqrt{25-\left(\sqrt{\frac{50}{3}}\right)^2}$$

$$= \frac{500}{3\sqrt{3}}\pi$$

$$= \frac{500\sqrt{3}}{9}\pi$$

$\therefore$ the maximum value of V is $\frac{500\sqrt{3}}{9}\pi\ \text{cm}^3$

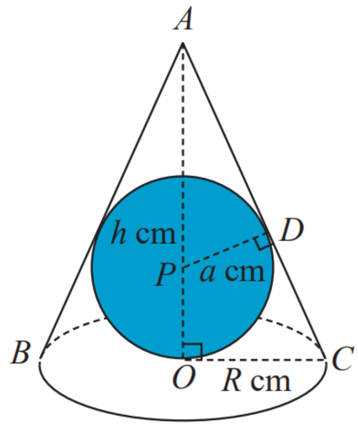
23
Solution

(a) Let centre of the sphere be P.

By Pythagoras Theorem

$$AC^2 = OA^2 + OC^2$$

$$AC^2 = h^2 + R^2 \text{ .... (1)}$$



Using similar triangles APD and ACO,

$$\frac{PA}{AC} = \frac{PD}{OC}$$

$$\frac{h-a}{AC} = \frac{a}{R}$$

$$AC = \frac{Rh - Ra}{a} \text{ .... (2)}$$

Substitute (1) into (2)

$$\left(\frac{Rh - Ra}{a}\right)^2 = h^2 + R^2$$

$$R^2h^2 - 2R^2ha + R^2a^2 = h^2a^2 + R^2a^2$$

$$R^2(h^2 - 2ha) = h^2a^2$$

$$\therefore\ R = \frac{ha}{\sqrt{(h^2 - 2ha)}} \quad \text{(Shown)}$$

(**Alternative Method**)

Let centre of the sphere be P.

$$AC^2 = h^2 + R^2 \text{ .... (3)}$$

$$AD = \sqrt{AP^2 - PD^2}$$

$$= \sqrt{((h-a)^2 - a^2)}$$

$$= \sqrt{h^2 - 2ha}$$

Using congruent triangles PCO and PCD, $CO = CD = R$ cm.

$$AC = AD + DC$$

$$= \sqrt{(h^2 - 2ha)} + R \text{ .... (4)}$$

Substitute (4) into (3)

$$(\sqrt{(h^2 - 2ha)} + R)^2 = h^2 + R^2$$

$$h^2 + R^2 = h^2 - 2ha + 2R\sqrt{(h^2 - 2ha)} + R^2$$

$$\therefore\ R = \frac{ha}{\sqrt{(h^2 - 2ha)}} \quad \text{(Shown) .... (5)}$$

(b) Volume of cone, $V = \frac{1}{3}\pi R^2 h$ .... (6)

Substitute (5) into (6)

$$= \frac{1}{3}\pi\left(\frac{ha}{\sqrt{(h^2-2ha)}}\right)^2 h$$

$$= \frac{1}{3}\pi a^2 \frac{h^3}{h^2-2ha}$$

$$= \frac{1}{3}\pi a^2 \frac{h^2}{h-2a} \quad \text{..........................} \ (7)$$

Differentiate (7) with respect to h

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi a^2 \left[\frac{2h(h-2a)-h^2}{(h-2a)^2}\right]$$

$$= \frac{1}{3}\pi a^2 \left[\frac{h^2-4ha}{(h-2a)^2}\right]$$

$$= \frac{1}{3}\pi a^2 \frac{h(h-4a)}{(h-2a)^2}$$

For minimum volume of the cone, $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$

$$\frac{1}{3}\pi a^2 \frac{h(h-4a)}{(h-2a)^2} = 0$$

$$\frac{h(h-4a)}{(h-2a)^2} = 0$$

$$h(h-4a) = 0$$

$$h = 4a \ \text{ or } \ h = 0 \ (\text{reject } h > 0)$$

Use First Derivative Test to show minimum

h	$(4a)^-$	$4a$	$(4a)^+$
Sign of $\frac{\mathrm{d}V}{\mathrm{d}h}$	$-$ve	0	+ve
Tangent	\	—	/

Thus, V is a minumum at $h = 4a$.

Substitute $h = 4a$ into (7)

$$\therefore \text{minimum } V = \frac{1}{3}\pi a^2 \frac{(4a)^2}{4a-2a}$$

$$= \frac{8}{3}\pi a^3 \text{ cm}^3$$

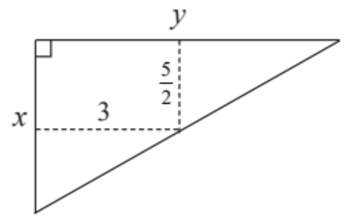
$\therefore$ the minimum volume of the cone in terms of a is $\frac{8}{3}\pi a^3$ cm^3.

Exercise 10

D Maxima and Minima Problems involving similar triangles

24
Solution

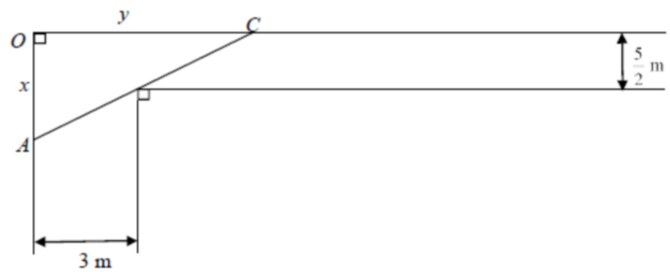
(a) Using similar triangles



$$\frac{y}{x} = \frac{3}{x - \frac{5}{2}}$$

$$y = \frac{6x}{2x-5} \text{ .... (1) (Shown)}$$

(b) Let the length of AC be L



By Pythagoras Theorem,

$$L^2 = x^2 + y^2 \text{ .... (2)}$$

Substitute (1) into (2)

$$L^2 = x^2 + \left(\frac{6x}{2x-5}\right)^2 \text{ .... (3)}$$

Differentiate (3) with respect to x

$$2L\frac{\mathrm{d}L}{\mathrm{d}x} = 2x + 2\left(\frac{6x}{2x-5}\right)\left[\frac{(2x-5)(6)-6x(2)}{(2x-5)^2}\right]$$

$$= \frac{2x(2x-5)^3 + 12x(-30)}{(2x-5)^2}$$

At minimum length of AC, let $\frac{\mathrm{d}L}{\mathrm{d}x} = 0.$

i.e. $$\frac{2x(2x-5)^3 + 12x(-30)}{(2x-5)^3} = 0$$

$$2x(2x-5)^3 - 360x = 0$$

$$2x\left[(2x-5)^3 - 180\right] = 0$$

$$x = \frac{5+\sqrt[3]{180}}{2} \text{ or } x = 0 \text{ (rejected } \because x > 0)$$

Use First Derivative test to show AC is minimum

x	$(5.3231)^-$	5.3231	$(5.3231)^+$
Sign of $\frac{\mathrm{d}L}{\mathrm{d}x}$	$-$ ve	0	$+$ ve

$\therefore$ AC is a minimum when $x = 5.3231$

Substitute $x = 5.3231$ into (3)

$$L^2 = x^2 + \left(\frac{6x}{2x-5}\right)^2$$

$$= \sqrt{(5.3231)^2 + \left[\frac{6(5.3231)}{2(5.3231)-5}\right]^2}$$

$$= 7.77$$

$\therefore$ the minimum length of AC is 7.77 m

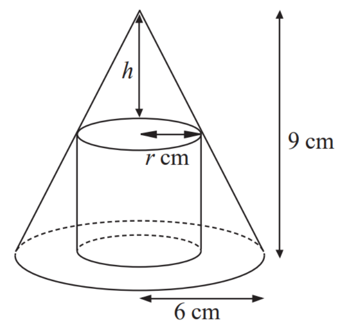
Since dimension of mirror = $5 < 7.767$

$\therefore$ the mover is able to carry the mirror through the corridors.

25
Solution

(a) Let h denote the leight of the cylinder in cm.

By similar triangles,



$$\frac{9}{9-h} = \frac{6}{r}$$

$$\frac{3r}{2} = 9-h$$

$$h = 9 - \frac{3r}{2} \text{ .... (1)}$$

Volume of the cylinder, $V = \pi r^2 h$ .... (2)

Substitute (1) into (2)

$$V = \pi r^2\left(9 - \frac{3r}{2}\right)$$

$$V = \frac{3\pi}{2}(6r^2 - r^3) \quad \text{(Shown) .... (3)}$$

(b) Differentiate (3) with respect to r

$$\frac{dV}{dr} = \frac{3\pi}{2}(12r - 3r^3)$$

For stationary value, $\frac{dV}{dr} = 0$

$$\therefore \quad 12r - 3r^2 = 0$$

$$3r(4-r) = 0$$

$$r = 0 \ \text{(Rejected, since } r > 0) \ \text{ or } \ r = 4$$

Use Second Derivative test to show V is maximum

$$\frac{d^2V}{dr^2} = \frac{3\pi}{2}(12-6r)$$

When $r = 4$,

$$\frac{d^2V}{dr^2} = \frac{3\pi}{2}[-12]$$

$$= -18\pi < 0$$

$\therefore$ maximum value of V occurs when $r = 4$

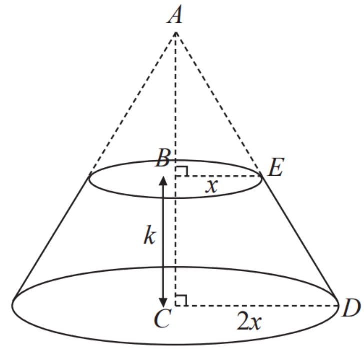
Substitute $r = 4$ into (3)

$$V = \pi(4)^2\left(9 - \frac{3(4)}{2}\right)$$

$$= 48\pi$$

$\therefore$ the maximum value of V is 48π cm^3

26
Solution



(a) By similar triangles,

$$\frac{BE}{CD}=\frac{AB}{AC}$$
$$\frac{x}{2x}=\frac{AB}{AB+k}$$
$$\frac{1}{2}=\frac{AB}{AB+k}$$
$$AB=k$$

Volume of the frustum, $V=\frac{1}{3}\pi(2x)^2 2k-\frac{1}{3}\pi(x)^2 k$

$$=\frac{7}{3}\pi x^2 k$$

Given that the volume of the frustum is 49 cm^3, $V=49$

$$49=\frac{7}{3}\pi x^2 k$$
$$k=\frac{21}{\pi x^2} \text{ .... (1)}$$

Let the slant height, AD be l

By observation, ΔACD is twice ΔABE.

$\therefore\ AE=\frac{1}{2}l$

By Pythagoras Theorem,

$$AD^2=(2x^2)+(2k)^2$$
$$l^2=(2x^2)+(2k)^2 \text{ .... (2)}$$

Total curved surface area

$$S=\pi(2x)(AD)-\pi x(AE)$$
$$S=\pi(2x)(l)-\pi x\left(\frac{1}{2}l\right)$$
$$=\frac{3}{2}\pi xl$$
$$S^2=\frac{9}{4}\pi^2x^2l^2 \text{ .... (3)}$$

Substitute (1) into (3)

$$S^2=\frac{9}{4}\pi^2x^2\left[(2x^2)+(2k)^2\right]$$

Substitute (2) into $S^2=\frac{9}{4}\pi^2x^2\left[(2x^2)+(2k)^2\right]$

$$=\frac{9}{4}\pi^2x^2\left[(2x^2)+\left(\frac{42}{\pi x^2}\right)^2\right]$$

$$S=\sqrt{9\pi^2x^4+\frac{3969}{x^2}} \quad \text{(Shown) .... (4)}$$

(b) Differentiate (4) with respect to x

$$2S\frac{\mathrm{d}S}{\mathrm{d}x}=36\pi^2x^3-\frac{7938}{x^3} \quad \text{..............................} \ (5)$$

For to be S minimum, $\frac{\mathrm{d}S}{\mathrm{d}x}=0$

i.e. $\quad 36\pi^2x^3-\frac{7938}{x^3}=0$

$$36\pi^2x^6-7938=0$$

Using GC, $x=1.67822 \ \ (x>0)$

Substitute $x=1.67822$ into (1)

$$k=\frac{21}{\pi(1.67822)}$$
$$=2.37340$$

Use Second Derivative test to show S is minimum

Differentiate (5) with respect to x

$$2\left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2+2S\frac{\mathrm{d}^2S}{\mathrm{d}x^2}=108\pi^2x^2+\frac{23814}{x^4}$$

When $x=1.67822$, $\frac{\mathrm{d}S}{\mathrm{d}x}=0$ and $S>0$

$$2S\frac{\mathrm{d}^2S}{\mathrm{d}x^2}=108\pi^2x^2+\frac{23814}{x^4}>0$$

$\therefore \ x=1.68$ and $k=2.37$ give minimum S.

Alternative Method (Use First Derivative test to show minimum)

x	$\left(\sqrt[6]{\frac{7938}{36\pi^2}}\right)^-$	$\sqrt[6]{\frac{7938}{36\pi^2}}$	$\left(\sqrt[6]{\frac{7938}{36\pi^2}}\right)^+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	-ve	0	+ve
Slope	\	—	/

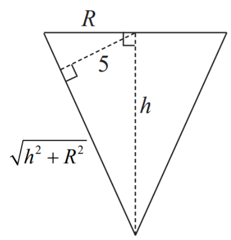
Therefore $x=1.678$ and $k=2.37$ gives minimum S.

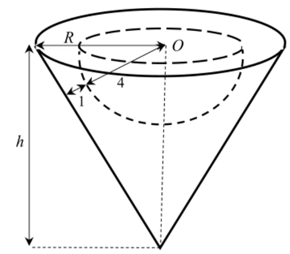
27
Solution

(a) Volume of the artwork, $V = \frac{1}{3}\pi R^2 h - \frac{2}{3}\pi(4)^3$

$$= \frac{1}{3}\pi R^2 h - \frac{128}{3}\pi \text{ .... (1)}$$

By similar triangles,





$$\frac{R}{\sqrt{h^2+R^2}} = \frac{5}{h}$$

$$h^2R^2 = 25(h^2+R^2)$$

$$R^2 = \frac{25h^2}{h^2-25} \text{ .... (2)}$$

Substitute (2) into (1)

$$\therefore \quad V = \frac{1}{3}\pi\left(\frac{25h^2}{h^2-25}\right)h - \frac{128}{3}\pi$$

$$= \frac{\pi}{3}\left(\frac{25h^3}{h^2-25}\right) - \frac{128\pi}{3} \text{ .... (3) (Shown)}$$

Differentiate (3) with respect to h

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{25\pi}{3}\left(\frac{(h^2-25)(3h^2)-h^3(2h)}{(h^2-25)^2}\right) - 0$$

$$= \frac{25\pi}{3}\left(\frac{h^2(h^2-75)}{(h^2-25)^2}\right) \text{ .... (4)}$$

For minimum volume of the artwork, let $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$.

i.e. $\frac{25\pi}{3}\left(\frac{h^2(h^2-75)}{(h^2-25)^2}\right) = 0$

$$h^2(h^2-75) = 0$$

Since $h \neq 0$, $\quad h^2 - 75 = 0$

$$h = \sqrt{75} \quad (\text{Since } h > 0)$$

$$= 5\sqrt{3}$$

Use First Derivative test to show V is minimum

	$h = (\sqrt{75})^-$	$h = (\sqrt{75})$	$h = (\sqrt{75})^+$
$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{25\pi}{3}\left(\frac{h^2(h^2-75)}{(h^2-25)^2}\right)$	$h^2 < 75$ $h^2 - 75 < 0$ $\frac{\mathrm{d}V}{\mathrm{d}h} < 0$	$h^2 = 75$ $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$	$h^2 > 75$ $h^2 - 75 > 0$ $\frac{\mathrm{d}V}{\mathrm{d}h} > 0$
gradient	\	—	/

$\therefore$ When $h = 5\sqrt{3}$, V is minimum.

Alternative Method (Use Second Derivative test to show V is minimum)

Differentiate (4) with respect to h

$$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = \frac{(h^2-25)(4h^3-150h)-2(h^2-25)(2h)(h^2)(h^2-75)}{(h^2-25)^4}$$

When $h = 5\sqrt{3}$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = 0.51962 > 0$

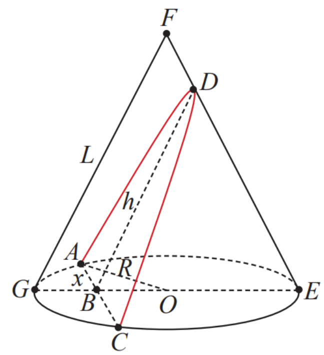
$\therefore$ when $h = 5\sqrt{3}$, V is minimum.

Substitute $h = 5\sqrt{3}$ into (3)

$$V = \frac{\pi}{3}\left(\frac{25(5\sqrt{3})^3}{(5\sqrt{3})^2-25}\right) - \frac{128\pi}{3}$$
$$= 206.046$$

$\therefore$ the minimum volume of the artwork is $206\ \text{cm}^3$ (3s.f.)

28
Solution



(a) By Pythagoras's theorem,

$$OB = \sqrt{OA^2 - AB^2}$$
$$= \sqrt{R^2 - x^2}$$

Using similar triangles,

$$\frac{BD}{GF} = \frac{BE}{GE}$$
$$\frac{h}{L} = \frac{R + \sqrt{R^2 - x^2}}{2R}$$
$$h = \frac{L}{2R}(R + \sqrt{R^2 - x^2}) \text{ .... (1)}$$

Given $S = \frac{4}{3}xh$ .... (2)

Substitute (1) into (2)

$$S = \frac{4}{3}x\left[\frac{L}{2R}(R + \sqrt{R^2 - x^2})\right]$$
$$= \frac{2LxR + \sqrt{R^2 - x^2}}{3R} \quad \text{(Shown)}$$

(b) Differentiate (3) with respect to x

$$\frac{dS}{dx} = \frac{2L}{3R}\left[x\left(\frac{-2x}{2\sqrt{R^2 - x^2}}\right) + \left(R + \sqrt{R^2 - x^2}\right)\right]$$
$$= \frac{2L}{3R}\left[\frac{-2x^2 + 2\sqrt{R^2 - x^2}(R + \sqrt{R^2 - x^2})}{2\sqrt{R^2 - x^2}}\right]$$

For stationary value, $\frac{dS}{dx} = 0$

i.e. $$\frac{2L}{3R}\left[\frac{-2x^2 + 2\sqrt{R^2 - x^2}(R + \sqrt{R^2 - x^2})}{2\sqrt{R^2 - x^2}}\right] = 0$$

$$-2x^2 + 2\sqrt{R^2 - x^2}(R + \sqrt{R^2 - x^2}) = 0$$
$$-2x^2 + 2R\sqrt{R^2 - x^2} + 2R^2 - 2x^2 = 0$$
$$2R^2 - 4x^2 + 2R\sqrt{R^2 - x^2} = 0$$
$$R\sqrt{R^2 - x^2} = 2x^2 - R^2$$
$$R^2(R^2 - x^2) = 4x^4 - 4x^2R^2 + R^4$$
$$4x^4 = 3x^2R^2$$
$$x = \frac{\sqrt{3}}{2}R \text{ (since } x > 0)$$

$$\frac{h}{x} = \frac{\frac{L}{2R}\left(R + \sqrt{R^2 - \frac{3}{4}R^2}\right)}{\frac{\sqrt{3}}{2}R}$$

$$= \frac{L}{2R}\left(R + \sqrt{R^2 - \frac{3}{4}R^2}\right)\left(\frac{2}{\sqrt{3}R}\right)$$

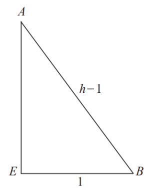
$$= \frac{L}{2R}\left(\frac{3R}{2}\right)\left(\frac{2}{\sqrt{3}R}\right)$$

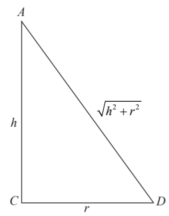
$$= \frac{\sqrt{3}L}{2R}$$

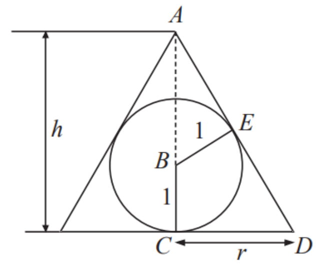
29
Solution

(a) By Pythagoras Theorem,

$$AB^2 = AE^2 + EB^2$$
$$AE = \sqrt{(h-1)^2 - 1}$$







Using similar triangles

$$\frac{AC}{AE} = \frac{CD}{EB}$$
$$\frac{h}{\sqrt{(h-1)^2 - 1}} = \frac{r}{1}$$
$$r = \frac{h}{\sqrt{(h^2 - 2h + 1) - 1}}$$
$$r^2 = \frac{h^2}{h^2 - 2h}$$
$$r^2 = \frac{h}{h-2} \quad \text{(Shown) .... (1)}$$

Let the volume of the cone be V

$$V = \frac{1}{3}\pi r^2 h \text{ .... (2)}$$

Substitute (1) into (2)

$$V = \frac{1}{3}\pi\left(\frac{h}{h-2}\right)h$$
$$= \frac{\pi h^2}{3(h-2)}$$

(b) Differentiate (3) with respect to h

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{3}\frac{(h-2)h - h^2(1)}{(h-2)^2}$$
$$= \frac{\pi h(h-4)}{3(h-2)^2}$$

For smallest volume of the cone, set $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$

i.e. $$\frac{\pi h(h-4)}{3(h-2)^2} = 0$$
$$h(h-4) = 0$$
$$h = 0 \text{ (Rejected, since } h > 0) \quad \text{or} \quad h = 4$$

Substitute $h = 4$ into (1)

$$r^2 = \frac{4}{4-2}$$
$$r = \sqrt{2}$$

The value of r is $\sqrt{2}$ and the value of h is 4

Use First Derivative test to show V is minimum

r	$\left(\sqrt{2}\right)^-$	$\left(\sqrt{2}\right)$	$\left(\sqrt{2}\right)^+$
$\frac{\mathrm{d}V}{\mathrm{d}h}$	$-ve$	0	$+ve$
Slope	\	—	/

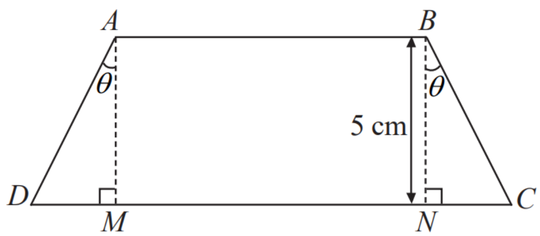
$\therefore$ V is the smallest when $r = \sqrt{2}$

Exercise 10

E Maxima and Minima Problems involving Trigonometric Functions

30
Solution

(a) Refer to the diagram.



Using trigonometric ratios,

$$\tan\theta = \frac{DM}{5}$$

$\therefore\ DM = 5\tan\theta$

$$\cos\theta = \frac{5}{AD}$$

$\therefore\ AD = 5\sec\theta$ .... (1)

Given $AD + AB + BC = 20$ .... (2)

Substitute (1) into (2)

$5\sec\theta + AB + 5\sec\theta = 20$

$\therefore\ AB = 20 - 10\sec\theta$

Area of trapezium $ABCD, A = \frac{1}{2}(BN)(AB + DC)$

$$= \frac{1}{2}(5)\left[(20-10\sec\theta)+(20-10\sec\theta)+2(5\tan\theta)\right]$$

$= 100 - 50\sec\theta + 25\tan\theta$ (Shown) .... (3)

(b) Differentiate (3) with respect to θ

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = -50\sec\theta\tan\theta + 25\sec^2\theta$$

For maximum area $ABCD$, let $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$

i.e. $-50\sec\theta\tan\theta + 25\sec^2\theta = 0$

$25\sec\theta(-2\tan\theta + \sec\theta) = 0$

$25\sec\theta = 0$ (no solution) or $2\tan\theta = \sec\theta$

$$\sin\theta = \frac{1}{2}$$

$$\therefore\ \theta = \frac{\pi}{6}$$

Substitute $\theta = \frac{\pi}{6}$ into (3)

$$A = 100 - 50\sec\left(\frac{\pi}{6}\right) + 25\tan\left(\frac{\pi}{6}\right)$$

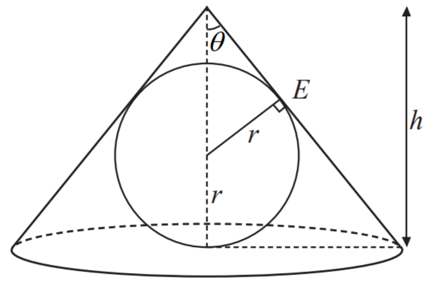
$$= 100 - 50\left(\frac{2}{\sqrt{3}}\right) + 25\left(\frac{1}{\sqrt{3}}\right)$$

$$= 100 - \frac{75}{\sqrt{3}}$$

$$= (100 - 25\sqrt{3})\ \text{cm}^2$$

$\therefore$ the maximum area of trapezium $ABCD$ is $(100 - 25\sqrt{3})\ \text{cm}^2$

31
Solution



(a) Let h = vertical height of cone.

$$h = BC + AC$$

$$\therefore \quad h = r + \frac{r}{\sin\theta}$$

$$= \frac{r}{\sin\theta}(1+\sin\theta) \quad \text{(Shown)} \text{ .... (1)}$$

(b) $\tan\theta = \dfrac{\text{Radius of the cone}}{h}$

$\therefore$ Radius of the cone $= h\tan\theta$

$\cos\theta = \dfrac{h}{\text{Slant height of the cone}}$

$\therefore$ Slant height of the cone $= \dfrac{h}{\cos\theta}$

Area of metal used, $A = \pi(\text{Radius of the cone})(\text{Slant height of the cone})$

$$= \pi(h\tan\theta)\left(\frac{h}{\cos\theta}\right)$$

$$= \pi(h)\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{h}{\cos\theta}\right)$$

$$= \pi h^2\left(\frac{\sin\theta}{\cos^2\theta}\right) \text{ .... (2)}$$

Substitute (1) into (2)

$$= \pi\left[\left(\frac{r}{\sin\theta}\right)(1+\sin\theta)\right]^2\left(\frac{\sin\theta}{\cos^2\theta}\right)$$

$$= \pi\left[\frac{r^2}{\sin^2\theta}(1+\sin\theta)^2\right]\left(\frac{\sin\theta}{\cos^2\theta}\right)$$

$$= \frac{\pi r^2(1+\sin\theta)^2}{\sin\theta\cos^2\theta}$$

$$= \frac{\pi r^2(1+\sin\theta)^2}{\sin\theta(1-\sin^2\theta)}$$

$$= \frac{\pi r^2(1+\sin\theta)(1+\sin\theta)}{\sin\theta(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{\pi r^2(1+\sin\theta)}{\sin\theta(1-\sin\theta)} \quad \text{(Shown)} \text{ .... (3)}$$

(c) Let $s = \sin\theta$

From (3): $A = \dfrac{\pi r^2(1+s)}{s(1-s)}$

$= \dfrac{\pi r^2(1+s)}{(s-s^2)}$

Differentiate A with respect to s

$$\frac{\mathrm{d}A}{\mathrm{d}s} = \pi r^2\left[\frac{s(1-s)(1)-(1+s)(1-2s)}{(s-s^2)^2}\right]$$

For A is minimum, let $\dfrac{\mathrm{d}A}{\mathrm{d}s} = 0$

$$\pi r^2\left[\frac{s(1-s)(1)-(1+s)(1-2s)}{(s-s^2)^2}\right] = 0$$

$$s(1-s)(1)-(1+s)(1-2s) = 0$$

$$s^2+2s-1 = 0$$

$$s = \frac{-2\pm\sqrt{4+4}}{2}$$

$$= -1+\sqrt{2} \text{ or } -1-\sqrt{2} \quad \text{(Rejected since } s>0\text{)}$$

Use First Derivative test to show A is minimum

x	0.4	$-1+\sqrt{2}$	0.5
$\dfrac{\mathrm{d}A}{\mathrm{d}x}$	$-$ve	0	$+$ve

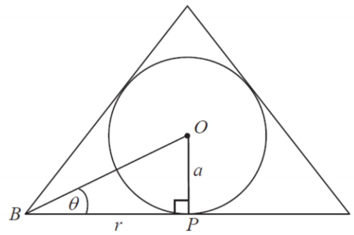
Therefore, A is minimum when $\sin\theta = s = -1+\sqrt{2}$ (Shown)

32
Solution

(a) From the diagram,
Using trigonometric ratios,

$$\cos\theta = \frac{BP}{OB}$$

$$\cos\theta = \frac{r}{\sqrt{r^2+a^2}}$$



$$\cos 2\theta = 2\cos^2\theta - 1$$

$$= 2\left(\frac{r}{\sqrt{r^2+a^2}}\right)^2 - 1$$

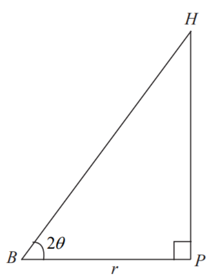
$$= \frac{2r^2}{r^2+a^2} - \frac{r^2+a^2}{r^2+a^2}$$

$$= \frac{r^2-a^2}{r^2+a^2} \qquad \text{(Shown)}$$

(b) Let T be the total surface area of the cone and the slant height of the cone be BH.
Using trigonometric ratios

$$\cos 2\theta = \frac{BP}{BH}$$

$$\therefore \qquad BH = \frac{r}{\cos 2\theta}$$



T = Curved surface area + Base area of the cone

$$T = \pi r(BH) + \pi(BP)^2$$

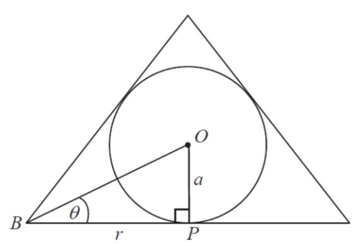
$$= \pi r\left(\frac{r}{\cos 2\theta}\right) + \pi r^2$$

$$= \pi r^2\left(\frac{1}{\cos 2\theta}\right) + \pi r^2$$

$$= \pi r^2\left(\frac{1}{\cos 2\theta} + 1\right)$$

$$= \pi r^2\left(\frac{r^2+a^2}{r^2-a^2} + 1\right)$$

$$= \frac{2\pi r^4}{r^2-a^2} \quad \text{............................ (1)}$$



Differentiate T with respect to r

$$\frac{\mathrm{d}T}{\mathrm{d}r}=\frac{8\pi r^3(r^2-a^2)-2\pi r^4(2r)}{(r^2-a^2)^2}$$
$$=\frac{4\pi r^3(r^2-2a^2)}{(r^2-a^2)^2}$$

For stationary, let $\frac{\mathrm{d}T}{\mathrm{d}r}=0$

$$\therefore \quad \frac{4\pi r^3(r^2-2a^2)}{(r^2-a^2)^2}=0$$
$$(r-\sqrt{2}a)(r+\sqrt{2}a)=0$$
$$r=\sqrt{2}a \quad \text{or} \quad r=-\sqrt{2}a \quad \text{(Rejected, since } r>0\text{)}$$

Use Second Derivative Test

$$\frac{\mathrm{d}^2T}{\mathrm{d}r^2}=4\pi\frac{(r^2-a^2)^2(5r^4-6a^2r^2)-r^3(r^2-2a^2)\left[2(r^2-a^2)(2r)\right]}{(r^2-a^2)^4}$$

When $r=\sqrt{2}a$,

$$=4\pi\frac{(a^2)^2(20a^4-12a^4)-0}{(a^2)^4}$$
$$=32\pi>0$$

Hence $r=\sqrt{2}a$ gives minimum T.

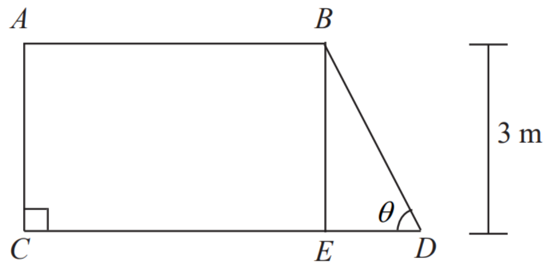
Substitute $r=\sqrt{2}a$ into (1)

$$T=\frac{2\pi(\sqrt{2}a)^4}{(\sqrt{2}a)^2-a^2}$$
$$=8\pi a^2$$

$\therefore$ the minimum T is $8\pi a^2$

33
Solution

(a) Refer to the diagram.



Using trigonometric ratios,

$$\sin\theta = \frac{3}{BD}$$

$$\therefore\ BD = \frac{3}{\sin\theta}$$

$$AB = AD - BD$$
$$= 8 - \frac{3}{\sin\theta}$$

Let the area of the flower bed be A.

A = Area of rectangle $ABEC$ + Area of triangle BED

$$= AB \times BE + \frac{1}{2} \times DE \times BE$$

$$= \left(8 - \frac{3}{\sin\theta}\right)(3) + \frac{1}{2}\left(\frac{3}{\sin\theta}\right)(3)$$

$$= 24 - 9\operatorname{cosec}\theta + \frac{9}{2}\cot\theta \ \text{............................ (1)} \quad \text{(Shown)}$$

(b) Differentiate (1) with respect to θ

$$A = 24 - 9\operatorname{cosec}\theta + \frac{9}{2}\cot\theta$$

$$\frac{dA}{d\theta} = 9\cot\theta\operatorname{cosec}\theta - \frac{9}{2}\operatorname{cosec}^2\theta$$

$$= \frac{9\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta} - \frac{9}{2\sin^2\theta}$$

$$= \frac{9\cos\theta}{\sin^2\theta} - \frac{9}{2\sin^2\theta}$$

$$= \frac{9}{2}\left(\frac{2\cos\theta - 1}{(\sin\theta)^2}\right)$$

For stationary, let $\frac{dA}{d\theta} = 0$,

$$\frac{9}{2}\left(\frac{2\cos\theta - 1}{(\sin\theta)^2}\right) = 0$$

$$2\cos\theta = 1$$

$$\theta = \frac{\pi}{3}$$

Use First Derivative test to show A is largest

θ	$\left(\frac{\pi}{3}\right)^-$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{3}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}\theta}$	$+$	0	$-$
	$\nearrow$	$\rightarrow$	$\searrow$

$\therefore$ the area is largest when $\theta = \frac{\pi}{3}$.

34
Solution

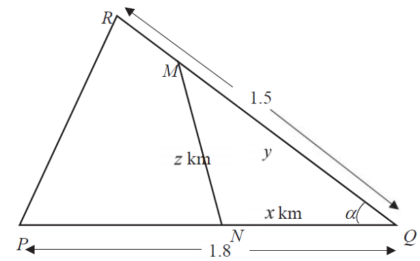
(a) Consider ΔMNO

Using cosine rule,

$$z^2 = x^2 + y^2 - 2xy\cos\alpha \text{ .... (1)}$$

Given that the residential and commercial developments have the same area size,

i.e. Area of the triangle MNQ = Area quadrilateral $RPNM$



$$\therefore \quad 2\left(\frac{1}{2}xy\sin\alpha\right) = \frac{1}{2}(1.5)(1.8)\sin\alpha$$

$$xy = 1.35 \text{ .... (2)}$$

$$y = \frac{1.35}{x} \text{ .... (3)}$$

Substitute (2) and (3) in (1):

$$z^2 = x^2 + \left(\frac{1.35}{x}\right)^2 - 2(1.35)\cos\alpha$$

$$z^2 = x^2 + \frac{1.8225}{x^2} - 2.7\cos\alpha \quad \text{(Shown) .... (4)}$$

(b) Differentiate (4) with respect to x,

$$2z\frac{dz}{dx} = 2x - \frac{3.645}{x^3}$$

For minimise construction costs, let $\frac{dz}{dx} = 0$

$$2z(0) = 2x - \frac{3.645}{x^3}$$

$$2x - \frac{3.645}{x^3} = 0$$

$$2x^4 - 3.645 = 0$$

$$x^4 = 1.8225$$

$$x = \sqrt[4]{1.8225} \quad \text{or} \quad x = -\sqrt[4]{1.8225} \quad \text{(Rejected since } x > 0\text{)}$$

$$= 1.16189$$

$$= 1.162 \text{ m (3 dp)}$$

Alternative solution

$$z = \left(x^2 + \frac{1.8225}{x^2} - 2.7\cos\alpha\right)^{\frac{1}{2}}$$

$$\frac{dz}{dx} = \frac{1}{2}\left(x^2 + \frac{1.8225}{x^2} - 2.7\cos\alpha\right)^{-\frac{1}{2}}\left(2x - \frac{2(1.8225)}{x^3}\right)$$

$$= \frac{1}{2\sqrt{x^2 + \frac{1.8225}{x^2} - 2.7\cos\alpha}}\left(2x - \frac{2(1.8225)}{x^3}\right)$$

For minimise construction costs, let $\frac{\mathrm{d}z}{\mathrm{d}x}=0$

For stationary values of z, $\frac{\mathrm{d}z}{\mathrm{d}x}=0$

$$\frac{1}{2\sqrt{x^2+\frac{1.8225}{x^2}-2.7\cos\alpha}}\left(2x-\frac{2(1.8225)}{x^3}\right)=0$$

$$2x-\frac{3.645}{x^3}=0$$

$$2x^4-3.645=0$$

$$2x^4=1.8225$$

$$x=\sqrt[4]{1.8225}=1.16189 \quad \text{or} \quad x=-\sqrt[4]{1.8225} \qquad (\text{Rejected since } x>0)$$

Use Second Derivative test to show z is minimum

Differentiate (4) with respect to x

$$2\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2+2z\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=2+\frac{10.935}{x^4}$$

For the value of z to be minimum, $\frac{\mathrm{d}z}{\mathrm{d}x}=0$

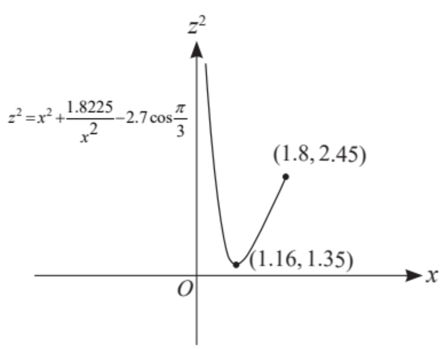
$$2z\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=2+\frac{10.935}{x^4}$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=\frac{1}{2z}\left(2+\frac{10.935}{x^4}\right)$$

For $x=\sqrt[4]{1.8225}$ and $z>0$, $\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=\frac{1}{2z}\left(2+\frac{10.935}{1.8225}\right)>0$

Hence the length of MN is a minimum when $x=1.162$ m (3 decimal places)

(c)



35
Solution

(a) Refer to the diagram.

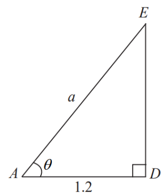
Using trigonometric ratios,

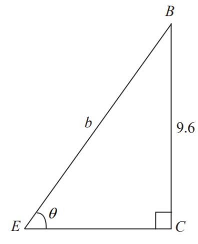
$$\cos\theta = \frac{1.2}{a}$$

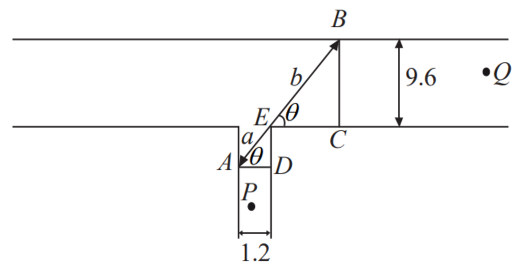
$$\therefore \ a = \frac{1.2}{\cos\theta}$$

$$\sin\theta = \frac{9.6}{b}$$

$$\therefore \ b = \frac{9.6}{\sin\theta}$$







$$AB = a + b$$

$$x = a + b$$

$$= \frac{1.2}{\cos\theta} + \frac{9.6}{\sin\theta}$$

$$= 1.2\sec\theta + 9.6\operatorname{cosec}\theta \quad \text{(Shown) .... (1)}$$

(b) Differentiate (4) with respect to x,

$$\frac{dx}{d\theta} = 1.2\sec\theta\tan\theta - 9.6\operatorname{cosec}\theta\cot\theta$$

For the length rod to be longest, $\frac{dx}{d\theta} = 0$.

$$1.2\sec\theta\tan\theta - 9.6\operatorname{cosec}\theta\cot\theta = 0$$

$$1.2\sec\theta\tan\theta = 9.6\operatorname{cosec}\theta\cot\theta$$

$$\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right) = 8\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\sin^3\theta = 8\cos^3\theta$$

$$\frac{\sin^3\theta}{\cos^3\theta} = 8$$

$$\tan\theta = (8)^{\frac{1}{3}}$$

$$= 2$$

$$\frac{d^2x}{d\theta^2} = 1.2\left[\sec\theta\tan^2\theta + \sec^3\theta\right] - 9.6\left[-\operatorname{cosec}\theta\cot^2\theta - \operatorname{cosec}^3\theta\right]$$

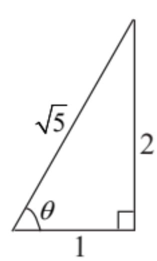
$$= 1.2\left[\sec\theta\tan^2\theta + \sec^3\theta\right] + 9.6\left[\operatorname{cosec}\theta\cot^2\theta\operatorname{cosec}^3\theta\right]$$

When $\theta = \tan^{-1}2$ which is acute,

$$\frac{d^2x}{d\theta^2} = 40.2 \ \text{(3 s.f.)} > 0$$

$\therefore x$ is minimum when $\theta = \tan^{-1}2$ rad.

Since $\tan\theta = 2$, $\sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$



Length of longest metal rod

$= 1.2\sec\theta + 9.6\operatorname{cosec}\theta$

$= 1.2(\sqrt{5}) + 9.6\left(\frac{\sqrt{5}}{2}\right)$

$= 6\sqrt{5}$ m

Note

Since x is minimum when $\theta = \tan^{-1} 2$ rad, the two walls and this distance will correspond to the maximum length of thin rod that can be carried horizontally into the room.

(c) From the diagram,

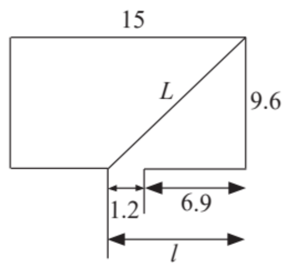
$l = 6.9 + 1.2 = 8.1$

By Pythagoras Theorem,

$L = \sqrt{8.1^2 + 9.6^2}$

$= \sqrt{157.77}$

$= 12.6$ m (3 s.f.)



The length of the longest thin rod that can be carried into this room is 12.6 m.

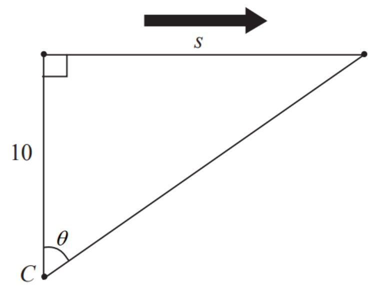
36
Solution

(a) Let x be distance that the car starts at point S run in t seconds.

Given that the car is travelling a constant speed of p m/s, $\frac{\mathrm{d}x}{\mathrm{d}t} = p$

Refer to the diagram.

$\tan\theta = \frac{x}{10}$ .... (1)



Differentiate θ with respect to t,

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{10}\frac{\mathrm{d}x}{\mathrm{d}t}$ .... (2)

When $x = 10$, substitute $x = 10$ into (1)

$$\tan\theta = \frac{10}{10}$$

$$\theta = \frac{\pi}{4}$$

Substitute $\theta = \frac{\pi}{4}$ into (2)

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{10}\frac{p}{\sec^2\left(\frac{\pi}{4}\right)}$$

$$= \frac{p}{10}\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{p}{20}$$

$\therefore$ θ is increasing at the rate of $\frac{p}{20}$ radians/s

(b) Given that Car A travels at a constant speed of $2q$ m/s

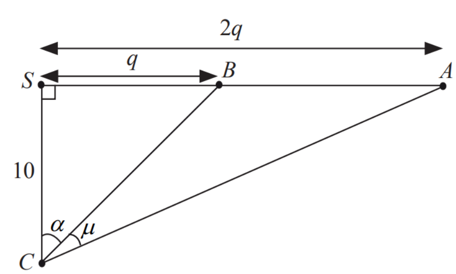
$\therefore$ the distance that the car A covers after T seconds $= 2qT$

Refer to the diagram.

$$\tan(\alpha + \mu) = \frac{2qT}{10}$$

$$= \frac{qT}{5}$$

$$\alpha + \mu = \tan^{-1}\left(\frac{qT}{5}\right)$$



Given that Car B travels at a constant speed of q m/s

$\therefore$ the distance that the car A covers after T seconds $= qT$

Refer to the diagram.

$$\tan\alpha = \frac{qT}{10}$$

$$\alpha = \tan^{-1}\left(\frac{qT}{10}\right)$$

Substitute (2) into (1)

$$\alpha + \mu = \tan^{-1}\left(\frac{qt}{5}\right)$$

$$\tan^{-1}\left(\frac{qt}{10}\right) + \mu = \tan^{-1}\left(\frac{qt}{5}\right)$$

$$\mu = \tan^{-1}\left(\frac{qt}{5}\right) - \tan^{-1}\left(\frac{qt}{10}\right) \qquad \text{(Shown) .... (3)}$$

(c) Differentiate (3) with respect to t,

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\frac{q}{5}}{1+\left(\frac{qt}{5}\right)^2} - \frac{\frac{q}{10}}{1+\left(\frac{qt}{10}\right)^2}$$

For maximum value of μ, let $\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$

i.e. $$\frac{q}{5}\left(1+\left(\frac{qt}{10}\right)^2\right) - \frac{q}{10}\left(1+\left(\frac{qt}{5}\right)^2\right) = 0$$

$$\frac{q}{5} + \frac{q^3}{500}t^2 - \frac{q}{10} - \frac{q^3}{250}t^2 = 0$$

$$\frac{q}{10} = \frac{q^3}{500}t^2$$

$$t = \sqrt{\frac{50}{q^2}}$$

$$= \frac{5\sqrt{2}}{q}$$

Substitute $t = \frac{\sqrt{50}}{q}$ into (1)

$$\mu = \tan^{-1}\left(\frac{q}{5}\cdot\frac{\sqrt{50}}{q}\right) - \tan^{-1}\left(\frac{q}{10}\cdot\frac{\sqrt{50}}{q}\right)$$

$$= \tan^{-1}\sqrt{2} - \tan^{-1}\frac{\sqrt{2}}{2}$$

$$= 0.340 \text{ radians}$$

Use Second Derivative test to show μ is maximum

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}t^2} = \frac{-\frac{2q^3t}{125}}{\left(1+\left(\frac{qt}{5}\right)^2\right)^2} - \frac{-\frac{2q^3t}{1000}}{\left(1+\left(\frac{qt}{10}\right)^2\right)^2}$$

When $t^2 = \dfrac{50}{q^2}$,

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}t^2} = \frac{-\dfrac{2q^3}{125}\left(\dfrac{\sqrt{50}}{q}\right)}{(1+2)^2} - \frac{-\dfrac{2q^3}{1000}\left(\dfrac{\sqrt{50}}{q}\right)}{\left(1+\dfrac{1}{2}\right)^2}$$

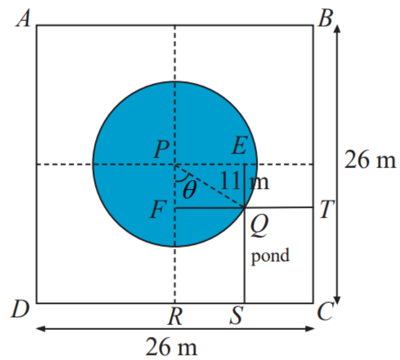
$$= 2q^2\sqrt{50}\left(\frac{-1}{125(9)} + \frac{4}{1000(9)}\right)$$

$$= -\frac{\sqrt{50}}{1125}q^2 < 0$$

Hence μ is a maximum when $t = \dfrac{5\sqrt{2}}{q}$.

37
Solution

(a) Refer to the diagram.



Using trigonometric ratios,

$$\sin\theta = \frac{FQ}{PQ}$$

$$\sin\theta = \frac{FQ}{11}$$

$$\therefore\ FQ = 11\sin\theta$$

$$\cos\theta = \frac{EQ}{PQ}$$

$$\cos\theta = \frac{EQ}{11}$$

$$\therefore\ EQ = 11\cos\theta$$

$$X = (FT - FQ)(ES - EQ)$$
$$= (13 - 11\sin\theta)(13 - 11\cos\theta) \text{ .... (1)}$$

Differentiate (1) with respect to θ,

$$\frac{\mathrm{d}X}{\mathrm{d}\theta} = (-11\cos\theta)(13 - 11\cos\theta) + (13 - 11\sin\theta)(11\sin\theta)$$
$$= -143\cos\theta + 121\cos^2\theta + 143\sin\theta - 121\sin^2\theta$$
$$= 143\sin\theta - 143\cos\theta + 121(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$$
$$= 143\sin\theta - 143\cos\theta - 121(\sin\theta - \cos\theta)(\cos\theta + \sin\theta)$$
$$= 11(13(\sin\theta - \cos\theta)) - 121(\sin\theta - \cos\theta)(\cos\theta + \sin\theta)$$
$$= 11(\sin\theta - \cos\theta)(13 - 11\sin\theta - 11\cos\theta) \quad \text{(Shown)}$$

(b) For stationary values of X, let $\frac{\mathrm{d}X}{\mathrm{d}\theta} = 0$.

i.e. $11(\sin\theta - \cos\theta)(13 - 11\sin\theta - 11\cos\theta) = 0$

$$(\sin\theta - \cos\theta)(13 - 11\sin\theta - 11\cos\theta) = 0$$

$$\sin\theta - \cos\theta = 0 \quad \text{or} \quad 13 - 11\sin\theta - 11\cos\theta = 0$$

$$\tan\theta = 1 \quad \text{or} \quad 11\sin\theta + 11\cos\theta = 13 \quad \triangleleft \text{ Using } R\text{ - formula}$$

$$\theta = \frac{\pi}{4} \qquad \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = \frac{13}{11}$$

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{13}{11\sqrt{2}}$$

$$\theta + \frac{\pi}{4} = 0.98935$$

$$\theta = 0.20396 \quad \text{since} \quad 0 \le \theta \le \frac{\pi}{4}$$

$$\therefore\ k = \frac{13}{11\sqrt{2}},\ \alpha = \frac{\pi}{4},\ \theta_1 = \frac{\pi}{4} \text{ and } \theta_2 = 0.20396$$

(c) Using first derivative test

θ	0.78	$\frac{\pi}{4}$	0.79
$\frac{dX}{d\theta}$	$0.215 > 0$	0	$-0.183 < 0$

Hence X is a maximum when $\theta_1 = \frac{\pi}{4}$

Using first derivative test

θ	0.203	0.20396	0.204
$\frac{dX}{d\theta}$	$-0.06997 < 0$	0	$0.00318 > 0$

Hence X is a minimum when $\theta_2 = 0.20396$

Alernative Method (Second Derivative Test)

$$\frac{dX}{d\theta} = -143\cos\theta + 121\cos^2\theta + 143\sin\theta - 121\sin^2\theta$$

$$\frac{d^2X}{d\theta^2} = 143\sin\theta - 242\cos\theta\sin\theta + 143\cos\theta - 242\sin\theta\cos\theta$$

$$= 143\sin\theta - 484\cos\theta\sin\theta + 143\cos\theta$$

$$\left.\frac{d^2X}{d\theta^2}\right|_{\theta=\frac{\pi}{4}} = -39.8 < 0$$

$$\left.\frac{d^2X}{d\theta^2}\right|_{\theta=0.20396} = 72.999 > 0$$

Hence X is a maximum when $\theta = \frac{\pi}{4}$ and X is a minimum when $\theta = 0.20396$.

To find the minimum area covered by grass, we have to use the maximum area X.

Smallest area covered by grass

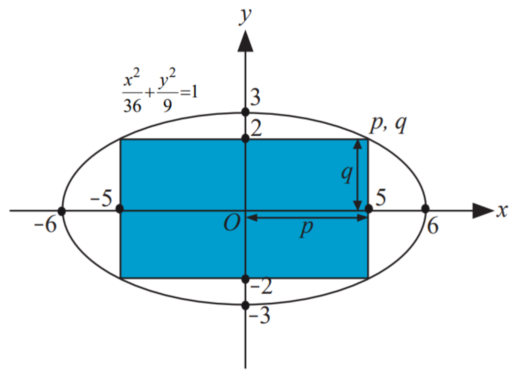
$= 676 - 4(27.3) - \pi(11)^2$

$= 187 \text{ m}^2$

Exercise 10

F Maxima and Minima Problems involving Conics

38
Solution



Given $\frac{x^2}{36}+\frac{y^2}{9}=1$ .... (1)

From the diagram, the point $(p,\ q)$ lies on the ellipse.

Substitute $(p,\ q)$ into (1)

$$\frac{p^2}{36}+\frac{q^2}{9}=1$$

$$p^2=36\left(1-\frac{q^2}{9}\right)$$

$$=36-4q^2 \text{ .... (2)}$$

Area of rectangle, $A=(2p)(2q)$ ◁ square both sides

$$A^2=\left[4pq\right]^2$$

$$A^2=16p^2q^2 \text{ .... (3)}$$

Substitute (2) into (3)

$$A^2=16(36-4q^2)q^2$$

$$=64q^2(9-q^2) \quad \text{(Shown) .... (4)}$$

Differentiate (4) with respect to q

$$2A\frac{\mathrm{d}A}{\mathrm{d}q}=1152q-256q^3 \quad \text{◁ differentiate implicity .... (5)}$$

For largest possible area of the rectangle, let $\frac{\mathrm{d}A}{\mathrm{d}q}=0$

i.e. $2A(0)=1152q-256q^3$

$$1152q-256q^3=0$$

$$q(1152-256q^2)=0$$

$$q=0 \text{ (rejected since } q>0) \text{ or } q^2=\frac{1152}{256}=\frac{9}{2}$$

$$q=\frac{3\sqrt{2}}{2} \quad \text{or} \quad q=-\frac{3\sqrt{2}}{2} \text{ (rejected, since } q>0)$$

Use Second Derivative test to verify largest possible area of the rectangle

Differentiate (4) with respect to q

$$(2A)\frac{\mathrm{d}^2 A}{\mathrm{d}q^2}+2\left(\frac{\mathrm{d}A}{\mathrm{d}q}\right)^2 = 1152-768q^2$$

When $\frac{\mathrm{d}A}{\mathrm{d}q}=0,\ q^2=\frac{9}{2}$,

$$(2A)\frac{\mathrm{d}^2 A}{\mathrm{d}q^2}+2(0)^2 = 1152-768\left(\frac{9}{2}\right)^2$$

$$\frac{\mathrm{d}^2 A}{\mathrm{d}q^2}=\frac{1152-768\left(\frac{9}{2}\right)^2}{2A}<0$$

$\therefore$ the rectangle is maximum when $q=\frac{3\sqrt{2}}{2}$

When $q=\frac{3\sqrt{2}}{2}$, substitute $q=\frac{3\sqrt{2}}{2}$ into (1)

$$p^2 = 36-4\left(\frac{9}{2}\right)$$

$$\begin{aligned} p &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

Hence the exact length and width of the largest possible area of the rectangle are $6\sqrt{2}$ and $3\sqrt{2}$ respectively.

39
Solution

Given $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right) \quad \text{..............................} \ (1)$$

Let the volume of the cylinder be V.

$$V=\pi y^2(2x)$$
$$=2\pi xy^2 \quad \text{..............................} \ (2)$$

Substitute (1) into (2)

$$V=2\pi b^2 x\left(1-\frac{x^2}{a^2}\right)$$
$$=2\pi b^2 x-\frac{2\pi b^2}{a^2}x^3 \quad \text{..............................} \ (3)$$

Differentiate (3) with respect to x

$$\frac{\mathrm{d}V}{\mathrm{d}x}=2\pi b^2-\frac{6\pi b^2}{a^2}x^2$$

For maximum volume of the cylinder, let $\dfrac{\mathrm{d}V}{\mathrm{d}x}=0$

$$2\pi b^2-\frac{6\pi b^2}{a^2}x^2=0$$
$$2\pi b^2\left(1-\frac{3}{a^2}x^2\right)=0$$
$$1-\frac{3}{a^2}x^2=0$$
$$x^2=\frac{a^2}{3}$$
$$x=\frac{a}{\sqrt{3}} \quad \text{or} \quad x=-\frac{a}{\sqrt{3}} \quad (\text{Rejected, since } x>0)$$

Use Second Derivative Test to show V is maximum

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2}=-\frac{12\pi b^2 x}{a^2}$$

For for all $x>0$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2}=-\dfrac{12\pi b^2 x}{a^2}<0$

Hence V is maximum for $x=\dfrac{a}{\sqrt{3}}$

Substitute $x = \frac{a}{\sqrt{3}}$ into (3)

$$V = \frac{2\pi ab^2}{\sqrt{3}} - \frac{2\pi ab^2}{3\sqrt{3}}$$
$$= \frac{4\pi ab^2}{3\sqrt{3}}$$

Therefore maximum volume of the cylinder is $\frac{4\pi ab^2}{3\sqrt{3}}$ units^2.

Ratio of the maximum volume of the cylinder to that of the volume enclosed by the ellipsoid

$$= \frac{4\pi ab^2}{3\sqrt{3}} : \frac{4\pi ab^2}{3}$$
$$= 1 : \sqrt{3}$$

Exercise 10

G Maxima and Minima Problems involving Costing

40
Solution

(a) Number of times that the manager needs to order in a year $= \dfrac{1200}{x}$

Total ordering cost $= 50\left(\dfrac{1200}{x}\right) = \dfrac{60000}{x}$

$C = \text{Storage Cost} + \text{Purchase Cost} + \text{Ordering Cost}$

$= 6x + 200(1200) + \dfrac{60000}{x}$

$\therefore$ the estimated total cost for the year, \$ C, is $C = 6x + 240000 + \dfrac{60000}{x}$. .... (1) (Shown)

(b) Differentiate (1) with respect to x

$\dfrac{\mathrm{d}C}{\mathrm{d}x} = 6 - \dfrac{60000}{x^2}$ .... (2)

When C is minimum, let $\dfrac{\mathrm{d}C}{\mathrm{d}x} = 0$

$$6 - \frac{60000}{x^2} = 0$$
$$\frac{60000}{x^2} = 6$$
$$x^2 = 10000$$
$$x = 100 \quad \text{or} \quad x = -100 \text{ (rejected since } x > 0)$$

Use Second Derivative Test to show C is minimum

Differentiate (2) with respect to x

$\dfrac{\mathrm{d}^2C}{\mathrm{d}x^2} = \dfrac{120000}{x^3}$

$\dfrac{\mathrm{d}^2C}{\mathrm{d}x^2} > 0$ for all $x > 0$

$\therefore$ C is minimum when $x = 100$

Substitute $x = 100$ into (1)

$C = 6(100) + 240000 + \dfrac{60000}{100} = \$241200$

Therefore minimum value of C is \$241200.

(c) This is not a reasonable model as various components of the cost differs depending on external factors such as the economical climate and real estate climate. For example the rental cost of warehouse will differs from year to year.

(d) Let y be the number of television sets sold.

Gradient of the line $= \dfrac{1200-0}{300-700} = -3$

Equantion of the line,

$$y - 1200 = -3(S - 300)$$
$$y = -3S + 900 + 1200$$
$$= -3S + 2100$$

$\therefore$ the equation of the line represents the number of television sets sold is $y = -3S + 2100$.

$P = \text{Total Selling Price} - \text{Total Cost}$
$= (\text{number units of television sets}) \times (\text{selling price}) - \text{Total Cost}$
$= yS - 24000$
$= (-3S + 2100)S - 240000$
$= 2100S - 3S^2 - 240000$ .... (1)

(e) Differentiate (1) with respect to S

$\dfrac{\mathrm{d}P}{\mathrm{d}S} = 2100 - 6S$ .... (2)

When P is stationary, $\dfrac{\mathrm{d}P}{\mathrm{d}S} = 0$

$$2100 - 6S = 0$$
$$S = 350$$

Differentiate (2) with respect to S

$\dfrac{\mathrm{d}^2P}{\mathrm{d}S^2} = -6 < 0$ (maximum)

Therefore when S is \$350, P is maximum.

41
Solution

Volume of the box $= (\text{Length}) \times (\text{width}) \times (\text{height})$

$= x \times x \times y$

Given that the volume of the box is 27000 cm^3,

$$27000 = x^2 y$$

$$\therefore \quad y = \frac{27000}{x^2} \text{ .... (1)}$$

Area of cardboard used, $A = 2x^2 + 2xy$ .... (2)

Substitute (1) into (2)

$$A = 2x^2 + 2x\left(\frac{27000}{x^2}\right)$$

$$A = 2x^2 + \frac{108000}{x}$$

Length of ribbon used, $L = 2x + 2y$ .... (3)

Substitute (1) into (3)

$$L = 2x + \frac{54000}{x^2}$$

Let C be the total cost of the cardboard and tape

$$C = \$0.02A + \$0.01L$$

$$= 0.02\left(2x^2 + \frac{108000}{x}\right) + 0.01\left(2x + \frac{54000}{x^2}\right)$$

$$= 0.04x^2 + \frac{2160}{x} + 0.02x + \frac{540}{x^2} \text{ .... (4)}$$

Differentiate (4) with respect to x

$$\frac{\mathrm{d}C}{\mathrm{d}x} = 0.08x - \frac{2160}{x^2} + 0.02 - \frac{1080}{x^3}$$

For minimum total cost, $\frac{\mathrm{d}C}{\mathrm{d}x} = 0$,

$$0.08x - \frac{2160}{x^2} + 0.02 - \frac{1080}{x^3} = 0$$

$$0.08x^4 + 0.02x^3 - 2160x - 1080 = 0$$

From GC, $x = 30.08$ or $x = -0.50$ (rejected since $x > 0$)

Substitute $x = 30.08$ into (1)

$$y = \frac{27000}{(30.08)^2}$$

$$= 29.84$$

Hence the values of x and y that yield a box with the least total cost of the cardboard and tape are $x = 30.08$ cm and 29.84 cm.

42
Solution

(a) Total cost for applying a protective film, $C = \$k(\text{flat surfaces}) + \$2k(\text{curved surfaces})$

$$= \left[k\pi(3r)^2\right] + \left[2\pi(3r)^2(2k) + 2\pi rh(2k)\right]$$

$$= 45\pi r^2 k + 4\pi rhk$$

$$h = \frac{C - 45\pi r^2 k}{4\pi rk} \text{ .... (1)}$$

Volume of the solid, $V = \frac{2}{3}\pi(3r)^3 - \pi r^2 h$ .... (2)

Substitute (1) into (2)

$$= 18\pi r^3 - \pi r^2\left(\frac{C}{4\pi rk} - \frac{45}{4}r\right)$$

$$V = \frac{117}{4}\pi r^3 - \frac{Cr}{4k}$$

Differentiate V with respect to r

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{117}{4}\pi(3r^2) - \frac{C}{4k}$$

At stationary value of V, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$

$$\frac{117}{4}\pi(3r^2) - \frac{C}{4k} = 0$$

$$r^2 = \frac{C}{351\pi k}$$

$$r = \sqrt{\frac{C}{351\pi k}} \quad \text{or} \quad r = -\sqrt{\frac{C}{351\pi k}} \quad \text{(Rejected, since } r > 0\text{)}$$

$\therefore\ r = \sqrt{\frac{C}{351\pi k}}$ when V is stationary.

(b) Since $h = 76.5r$ does not satisfy $h < 3r$, $\therefore$ it is not possible for this solid to have a stationary value of V.

43
Solution

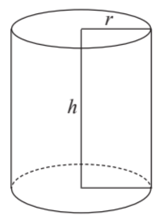
Let r cm, h cm, V cm^3 and A cm^2 denote the radius, the height, volume and total surface area of a cylindrical tin can respectively.

$$V = \pi r^2 h$$

Given that the volume of the can is 300 cm^3, $V = 300$.

$$\therefore \quad 300 = \pi r^2 h$$

$$h = \frac{300}{\pi r^2} \quad \text{.............................} \ (1)$$



$$A = 2\pi r^2 + 2\pi r h$$

Substitute (1) into (2)

$$A = 2\pi r^2 + 2\pi r\left(\frac{300}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{600}{r} \quad \text{.............................} \ (2)$$

Differentiate (2) with respect to r

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r - \frac{600}{r^2}$$

For least value of A, let $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$

$$4\pi r - \frac{600}{r^2} = 0$$

$$r^3 = \frac{150}{\pi}$$

$$r = \sqrt[3]{\frac{150}{\pi}}$$

Substitute $r = \sqrt[3]{\frac{150}{\pi}}$ into (2)

$$A = 2\pi\left(\frac{150}{\pi}\right)^{\frac{2}{3}} + \frac{600}{\left(\frac{150}{\pi}\right)^{\frac{1}{3}}}$$

$$= 2\pi\left(\frac{150}{\pi}\right)^{\frac{2}{3}} + 600\left(\frac{150}{\pi}\right)^{-\frac{1}{3}}$$

$$= \left(\frac{150}{\pi}\right)^{-\frac{1}{3}}\left(2\pi \times \left(\frac{150}{\pi}\right) + 600\right)$$

$$= 900\left(\frac{\pi}{150}\right)^{\frac{1}{3}}$$

Use Second Derivative Test to show the cost is minimum

$$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = 4\pi + \frac{1200}{r^3}$$

When $r = \sqrt[3]{\frac{150}{\pi}}$,

$$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = 4\pi + \frac{1200}{\left(\frac{150}{\pi}\right)}$$

$$= 12\pi > 0$$

Since A is minimum when $r = \sqrt[3]{\frac{150}{\pi}}$. Hence the cost of producing a can is also minimum.

$\therefore$ the least cost (in cents) of producing a can is $900\left(\frac{\pi}{150}\right)^{\frac{1}{3}}$, where $a = 900$, $b = 150$ and $k = \frac{1}{3}$

H Maxima and Minima Problems involving Speeds, Time and Distance

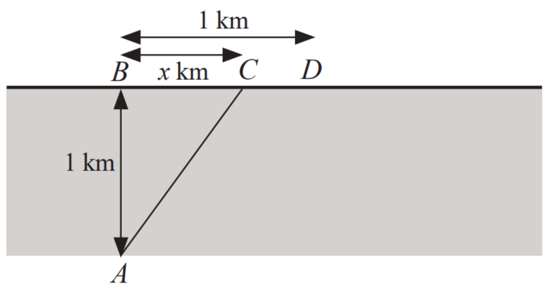
44
Solution

(a) Given T refers to the total time taken in hours by the aquathlon to get from A to C and then to D.

Refer to the diagram.
By Pythagoras Theorem,

$$AB^2 + BC^2 = AC^2$$
$$1^2 + x^2 = AC^2$$
$$AC = \sqrt{1+x^2}$$

$\therefore$ the distance AC is $\sqrt{1+x^2}$.



Time taken from A to C

$$= \frac{\text{Distance } AC}{\text{swiming speed}}$$

$$= \frac{\sqrt{1+x^2}}{4}$$

$$= \frac{1}{4}\sqrt{1+x^2} \text{ hours}$$

Refer to the diagram.

$$CD = BD - BC$$
$$CD = 1 - x$$

$\therefore$ the distance CD is $\sqrt{1+x^2}$.

Time taken from C to D

$$= \frac{\text{Distance } CD}{\text{Running speed}}$$

$$= \frac{1-x}{6}$$

$$= \frac{1}{6}(1-x) \text{ hours}$$

Hence, $T = (\text{Time taken from } A \text{ to } C) + (\text{Time taken from } C \text{ to } D)$

$$T = \frac{\sqrt{1+x^2}}{4} + \frac{(1-x)}{6} \quad \text{(Shown) .... (1)}$$

(b) Differentiate (1) with respect to x

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)(1+x^2)^{-\frac{1}{2}}(2x) - \frac{1}{6}$$

$$= \frac{x}{4\sqrt{1+x^2}} - \frac{1}{6}$$

For the time taken to be minimum, let $\frac{\mathrm{d}t}{\mathrm{d}x} = 0$

i.e. $$\frac{x}{4\sqrt{1+x^2}} - \frac{1}{6} = 0$$

$$6x = 4\sqrt{1+x^2}$$

$$36x^2 = 16 + 16x^2$$

$$20x^2 - 16 = 0 \text{ .... (2)}$$

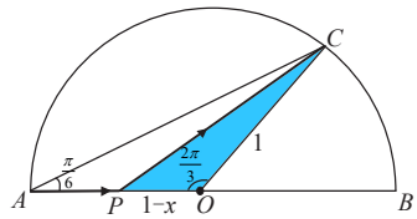
$\therefore\ a = 20,\ b = 0$ and $c = -16$

From (2): $20x^2 = 16$

$$x = \sqrt{\frac{4}{5}} \text{ (since } x \geq 0)$$

$\therefore$ the minimum value of x is $\sqrt{\frac{4}{5}}$.

45
Solution



(a) Refer to ΔAOP

Since $OA = OP$ (radius)

$$\therefore \angle COA = \pi - 2\left(\frac{\pi}{2}\right)$$
$$= \frac{2\pi}{3}$$

Consider triangle COP and using cosine rule,

$$CP^2 = OP^2 + OC^2 - 2(OP)(OC)\cos\angle COA$$
$$= (1-x)^2 + 1 - 2(1-x)(1)\cos\frac{2\pi}{3}$$
$$= 1 - 2x + x^2 + 1 - 2(1-x)\left(-\frac{1}{2}\right)$$
$$= 3 - 3x + x^2$$
$$CP = \sqrt{3-3x+x^2} \quad (\text{since } CP > 0) \quad (\text{Shown})$$

(b) Time taken to travel from A to P and then from P to C

$$T = \frac{\text{Distance } AP}{\text{speed on the road}} + \frac{\text{Distance } CP}{\text{speed on the rough ground}}$$
$$T = \frac{x}{5} + \frac{\sqrt{3-3x+x^2}}{3} \quad \text{.............................. (1)}$$

Differentiate (1) with respect to x

$$\frac{dT}{dx} = \frac{1}{5} + \frac{1}{6}\frac{2x-3}{\sqrt{3-3x+x^2}}$$

For minimum value of T, $\frac{dT}{dx} = 0$

i.e. $\frac{1}{5} + \frac{1}{6}\frac{2x-3}{\sqrt{3-3x+x^2}} = 0$

$$-5(2x-3) = 6\sqrt{3-3x+x^2}$$
$$25(4x^2 - 12x + 9) = 36(3 - 3x + x^2)$$
$$64x^2 - 192x + 117 = 0$$

Solving by GC: $x = 0.85048$ or $x = 2.1495$ (reject since $0 < x < 1$)

$= 0.850$ (correct to 3 s.f.)

Using first derivative test to show T is minimum.

x	0.85048^-	0.85048	0.85048^+
$\dfrac{dT}{dx}$	$-$ ve	0	+ ve
Slope	\	—	/

$\therefore$ T is minimum when $x = 0.850$ km.

Substitute $x = 0.850$ into (1)

$$T = \frac{0.85048}{5} + \frac{\sqrt{3 - 3(0.852048) + (0.85048)^2}}{3}$$
$$= 0.531 \text{ (3 s.f.)}$$

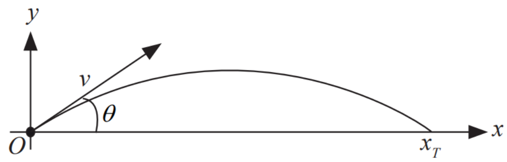
$\therefore$ the minimum value of T is 0.531 hours.

Exercise 10

I Maxima and Minima Problems involving Parametric Equations

46
Solution

(a) When the projectile to hits x_T, $y=0$



$$(v\sin\theta)t-\frac{1}{2}gt^2=0$$

$$t\left(v\sin\theta-\frac{1}{2}gt\right)=0$$

$t=0$ (rejected as $t>0$) or $v\sin\theta-\frac{1}{2}gt=0$

$$t=\frac{2v\sin\theta}{g}$$

(b) $A=\int_0^{x_T} y\ \mathrm{d}x$

$$=\int_0^{\frac{2v\sin\theta}{g}} y\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$$

$$=\int_0^{\frac{2v\sin\theta}{g}}\left((v\sin\theta)t-\frac{1}{2}gt^2\right)(v\cos\theta)\mathrm{d}t$$

$$=(v\cos\theta)\int_0^{\frac{2v\sin\theta}{g}}\left((v\sin\theta)t-\frac{1}{2}gt^2\right)\mathrm{d}t$$

$$=(v\cos\theta)\left[(v\sin\theta)\frac{t^2}{2}-\frac{1}{2}g\left(\frac{t^3}{3}\right)\right]_0^{\frac{2v\sin\theta}{g}}$$

$$=(v\cos\theta)\left(\frac{(v\sin\theta)}{2}\left(\frac{2v\sin\theta}{g}\right)^2-\frac{1}{6}g\left(\frac{2v\sin\theta}{g}\right)^3\right)$$

$$=(v\cos\theta)\left(\frac{2(v\sin\theta)^3}{g^2}-\frac{1}{6g^2}(8)(v\sin\theta)^3\right)$$

$$=(v\cos\theta)\left(\frac{2}{3}\frac{(v\sin\theta)^3}{g^2}\right)$$

$$=\frac{2v^4\sin^3\theta\cos\theta}{3g^2}\quad\text{(Shown) .... (1)}$$

(c) Differentiate (1) with respect to θ

$$\frac{\mathrm{d}A}{\mathrm{d}\theta}=\frac{2v^4}{3g^2}(\sin^3\theta(-\sin\theta)+\cos\theta(3\sin^2\theta\cos\theta))$$

$$=\frac{2v^4}{3g^2}(3\sin^2\theta\cos^2\theta-\sin^4\theta)$$

For maximum value of A, let $\dfrac{\mathrm{d}A}{\mathrm{d}\theta} = 0$

$$\frac{2v^4}{3g^2}(3\sin^2\theta\cos^2\theta - \sin^4\theta) = 0$$

$$\sin^2\theta(3\cos^2\theta - \sin^2\theta) = 0$$

$$3\cos^2\theta - \sin^2\theta = 0 \quad \text{or} \quad \sin^2\theta = 0$$

$$\tan^2\theta = 3 \quad \text{or} \quad \sin\theta = 0$$

$$\tan\theta = \pm\sqrt{3} \quad \text{or} \quad \theta = 0 \quad (\text{rejected } \theta > 0)$$

$$\tan\theta = \sqrt{3} \quad \text{or} \quad \tan\theta = -\sqrt{3}$$

$$\theta = \frac{\pi}{3} \qquad (\text{rejected as } \theta \text{ is acute, so } \tan\theta > 0)$$

Substitute $\theta = \dfrac{\pi}{3}$ into (1)

$$A = \frac{2v^4\sin^3\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)}{3g^2}$$

$$= \frac{2v^4\left(\left(\frac{\sqrt{3}}{2}\right)^3\right)\left(\frac{1}{2}\right)}{3g^2}$$

$$= \frac{v^4\sqrt{3}}{8g^2}$$

From (2):

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{2v^4}{3g^2}(3\sin^2\theta\cos^2\theta - \sin^4\theta)$$

$$= \frac{2v^4}{3g^2}\left(\frac{3}{4}(4\sin^2\theta\cos^2\theta) - \sin^4\theta\right)$$

$$= \frac{2v^4}{3g^2}\left(\frac{3}{4}(\sin 2\theta)^2 - \sin^4\theta\right)$$

Differentiate $\dfrac{\mathrm{d}A}{\mathrm{d}\theta}$ with respect to θ

$$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = \frac{2v^4}{3g^2}\left(\frac{3}{4}(4\sin 2\theta\cos 2\theta) - 4\sin^3\theta\cos\theta\right)$$

$$= \frac{2v^4}{3g^2}\left(\frac{3}{2}(\sin 4\theta) - 4\sin^3\theta\cos\theta\right)$$

When $\theta = \dfrac{\pi}{3}$

$$= \frac{2v^4}{3g^2}\left(\frac{3}{2}(\sin\frac{4\pi}{3}) - 4\sin^3\frac{\pi}{3}\cos\frac{\pi}{3}\right)$$

$$= \frac{2v^4}{3g^2}\left(\frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right)^3\frac{1}{2}\right)$$

$$= \frac{2v^4}{3g^2}\left(\frac{3\sqrt{3}}{2}\right)$$

$$= -\frac{\sqrt{3}v^4}{g^2}$$

Hence maximum $A = \dfrac{\sqrt{3}v^4}{8g^2}$ when $\theta = \dfrac{\pi}{3}$

47
Solution

(a) Given $x = 28(\theta\cos\theta + 1)$ .... (1)

and $y = 20(2 - \theta\sin\theta)$ .... (2)

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 28(\cos\theta - \theta\sin\theta)$$

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 20(-(\sin\theta + \theta\cos\theta))$$
$$= -20(\sin\theta + \theta\cos\theta)$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$
$$= \frac{-5(\sin\theta + \theta\cos\theta)}{7(\cos\theta - \theta\sin\theta)}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{5(\sin\theta + \theta\cos\theta)}{7(\cos\theta - \theta\sin\theta)}$$

(b)(i) Given that the tangents to the curve is parallel to the y-axis at C and D, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} = \infty.$

Let denominator of the $\frac{\mathrm{d}y}{\mathrm{d}x}$ be zero.

$\therefore \quad 7(\cos\theta - \theta\sin\theta) = 0$

$\cos\theta = \theta\sin\theta$

$\theta = -0.86033 \quad$ or $\quad 0.86033$

$\therefore\ \theta = -0.86,\ 0.86$ (correct to 2 significant figures)

(ii) Substitute $\theta = -0.86033$ into (1)

$x = 12.289$

$\therefore$ x-coordinate of $C = 12.289$

Substitute $\theta = 0.86033$ into (1)

$x = 43.710$

$\therefore$ x-coordinate of $D = 43.710$

Distance $CD = 43.710 - 12.289$

$= 31.4216$

Greatest width, CD, of the loop $= 31.4$ m (correct to 3 s.f)

(c)(i) Given $x = 28$, substitute $x = 28$ into (1)

$$28(\theta\cos\theta + 1) = 28$$
$$\theta\cos\theta + 1 = 1$$
$$\theta\cos\theta = 0$$
$$\theta = 0 \quad \text{or} \quad \cos\theta = 0$$
$$\theta = \pm\frac{\pi}{2}, \text{ where } |\theta| \le \frac{2\pi}{3}$$

$\therefore$ the value of θ at point A is 0.

(c)(ii) The 2 possible values of θ at B are $\pm\dfrac{\pi}{2}$

(c)(iii) Substitute $\theta = 0$ into (2)

$$y = 20(2 - 0\sin 0)$$
$$= 40$$

The y-coordinate at A is 40

Substitute $\theta = \dfrac{\pi}{2}$ into (2)

$$y = 20\left(2 - \frac{\pi}{2}\sin\frac{\pi}{2}\right)$$
$$= 8.584$$

The y-coordinate at B is 8.584

Length $AB = 40 - 8.584 = 31.4159$

$\therefore$ the distance $AB = 31.4$ m (correct to 3 s.f)

(d) From **(b)(i)** $\theta = 0.86033$, substitute $\theta = 0.86033$ into (2)

$$y = 20(2 - 0.86033\sin 0.86033)$$
$$= 26.956$$

The y-coordinate at E is 26.956

$$AE = 40 - 26.956$$
$$= 13.044$$

$$\frac{1}{2}CD = 15.7108$$
$$1.2 \times \frac{1}{2}CD = 18.85296$$
$$0.8 \times \frac{1}{2}CD = 12.56864$$

$\therefore$ the safety range is $(12.57, 18.85)$

Since AE is within the safety range, the design satisfies the condition.

Exercise 10

K Higher Order Questions

48
Solution

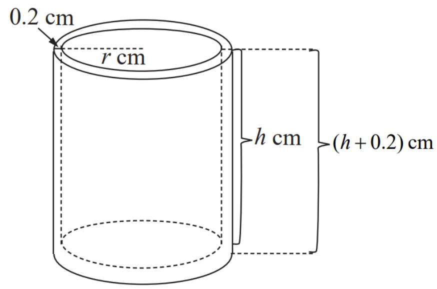
(a) Surface area of the rim

$= (\text{Area of big circle}) - (\text{Area of small circle})$

$= \pi\left(r + \frac{1}{5}\right)^2 - \pi r^2$

$= \pi\left(r^2 + \frac{2}{5}r + \frac{1}{25} - r^2\right)$

$= \frac{1}{25}\pi(10r + 1)$ (Shown)



(b) Let the volume of the can be V.

$V = \pi r^2 h$

Given that the inner capacity of the canister is 150π cm^2, i.e. $V = 150\pi$

$150\pi = \pi r^2 h$

$h = \frac{150}{r^2}$ .... (1)

Total surface area of the canister, A

$= (\text{Surface area of the rim}) + (\text{Area of the base of the canister}) + (\text{Area of the inner base of the canister})$
$+ (\text{Area of the inner curved surface of the canister}) + (\text{Area of the outer curved surface of the canister})$

$= \frac{1}{25}\pi(10r+1) + \pi r^2 + \pi\left(r + \frac{1}{5}\right)^2 + 2\pi rh + 2\pi\left(r + \frac{1}{5}\right)\left(h + \frac{1}{5}\right)$

$= \frac{2}{5}\pi r + \frac{\pi}{25} + \pi r^2 + \pi\left(r^2 + \frac{2}{5}r + \frac{1}{25}\right) + 2\pi rh + 2\pi\left(rh + \frac{1}{5}r + \frac{1}{5}h + \frac{1}{25}\right)$ .... (2)

Substitute (1) into (2)

$= \frac{2}{5}\pi r + \frac{\pi}{25} + \pi r^2 + \pi\left(r^2 + \frac{2}{5}r + \frac{1}{25}\right) + 2\pi r\left(\frac{150}{r^2}\right) + 2\pi\left[r\left(\frac{150}{r^2}\right) + \frac{1}{5}r + \frac{1}{5}\left(\frac{150}{r^2}\right) + \frac{1}{25}\right]$

$= \frac{2}{5}\pi r + \frac{\pi}{25} + \pi r^2 + \pi r^2 + \frac{2}{5}\pi r + \frac{\pi}{25} + 2\pi r\left(\frac{150}{r^2}\right) + 2\pi\left[r\left(\frac{150}{r^2}\right) + \frac{1}{5}r + \frac{1}{5}\left(\frac{150}{r^2}\right) + \frac{1}{25}\right]$

$= 2\pi\left(r^2 + \frac{2}{5}r + \frac{1}{25}\right) + 2\pi\left(\frac{150}{r}\right) + 2\pi\left(r\left(\frac{150}{r^2}\right) + \frac{1}{5}r + \frac{1}{5}\left(\frac{150}{r^2}\right) + \frac{1}{25}\right)$

$= 2\pi\left(r^2 + \frac{2}{5}r + \frac{1}{25}\right) + 2\pi\left(\frac{150}{r}\right) + 2\pi\left(\frac{150}{r} + \frac{1}{5}r + \frac{30}{r^2} + \frac{1}{25}\right)$

$= 2\pi\left(r^2 + \frac{3}{5}r + \frac{2}{25} + \frac{300}{r} + \frac{30}{r^2}\right)$ (Shown) .... (3)

(c) Differentiate (3) with respect to r

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi\left(2r + \frac{3}{5} - \frac{300}{r^2} - \frac{60}{r^3}\right) \text{ .... (4)}$$

For minimum value of A, let $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$.

i.e. $2\pi\left(2r + \frac{3}{5} - \frac{300}{r^2} - \frac{60}{r^3}\right) = 0$

$$2r + \frac{3}{5} - \frac{300}{r^2} - \frac{60}{r^3} = 0$$

By GC, $r = 5.2814$ $(\because r > 0)$

Use First Derivative Test to show A is minimum

r	5.2814^+	5.2814	5.2814^-
$\frac{\mathrm{d}A}{\mathrm{d}r}$	< 0	0	> 0
Slope	\	—	/

Hence A is minimum when $r = 5.2814$

(**Alternative Method**) Use Second Derivative Test to show minimum

Differentiate (4) with respect to r

$$\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = \pi\left(2 + \frac{600}{r^3} + \frac{180}{r^4}\right)$$

When $r = 5.2814$, $\frac{\mathrm{d}^2A}{\mathrm{d}r^2} = \pi\left(2 + \frac{600}{r^3} + \frac{180}{r^4}\right) = 39.61 > 0$

Therefore, A is minimum when $r = 5.2814$.

Substitute $r = 5.2814$ into (3)

$$A = 2\pi\left(r^2 + \frac{3}{5}r + \frac{2}{25} + \frac{300}{r} + \frac{30}{r}\right)$$
$$= 559.33$$
$$= 559 \text{ (3s.f.)}$$

$\therefore$ the minimum A is 559 cm^3

(d) Let x be the depth of water in the canister and volume of water in the canister be V.

$$V = \pi r^2 x \text{ .... (5)}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \pi r^2 \text{ (} r \text{ is constant)}$$

Given $h = 2r$, substitute $h = 2r$ into (5)

$$V = \pi r^2(2r)$$
$$V = 2\pi r^3$$

Given that the inner capacity of the can is 150π cm^2, i.e. $V = 150\pi$

$$150\pi = 2\pi r^3$$
$$r = \sqrt[3]{75}$$

Using Chain Rule,

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$$

Given that the volume of water in the canister is decreasing at a constant rate of $0.5\ \mathrm{cm}^3$ per second, i.e. $\frac{\mathrm{d}V}{\mathrm{d}t}=-0.5$

$$-0.5=\pi r^2\times\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{-0.5}{\pi r^2}$$

When $r=\sqrt[3]{75}$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{-0.5}{\pi\left(\sqrt[3]{75}\right)^2}$$

$$=-0.00895\ \mathrm{cms}^{-1}$$

The rate of decrease of the depth of water is $0.00895\ \mathrm{cms}^{-1}$.

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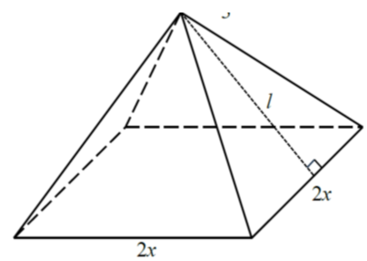
Solution



(a) $S = (\text{Base area of the square pyramid}) + 4(\text{Area of the side of the pyramid})$

$$= (2x)^2 + 4\times\frac{1}{2}(2x)(l)$$

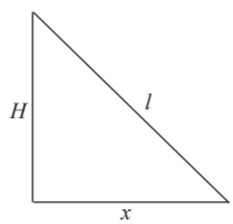
Given that the total external surface area of the candle is 144 cm^2, i.e. $S = 144$.

$$144 = (2x)^2 + 4\times\frac{1}{2}(2x)(l)$$

$$l = \frac{36}{x} - x \text{ .... (1)}$$

Let H be the height of the square pyramid

Use Pythagoras Theorem,



$$H = \sqrt{l^2 - x^2} \text{ .... (2)}$$

$$V = \frac{1}{3}(2x)^2 H \text{ .... (3)}$$

Substitute (1) into (3)

$$= \frac{4x^2}{3}\sqrt{l^2 - x^2}$$

Substitute (1) into $V = \frac{4x^2}{3}\sqrt{l^2 - x^2}$

$$V = \frac{1}{3}(2x)^2\sqrt{\left(\frac{36}{x} - x\right)^2 - x^2}$$

$$= \frac{1}{3}(4x^2)\sqrt{\left(\frac{36}{x}\right)\left(\frac{36}{x} - 2x\right)}$$

$$= \frac{1}{3}(4)(6)\sqrt{(x^2)^2\left(\frac{36}{x^2} - 2\right)}$$

$$= 8\sqrt{36x^2 - 2x^4} \text{ .... (4)} \quad \text{(Shown)}$$

(b) Differentiate (4) with respect to x

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 8\left(\frac{1}{2}\right)\frac{1}{\sqrt{36x^2 - 2x^4}}(72x - 8x^3)$$

$$= \frac{32x(9 - x^2)}{\sqrt{36x^2 - 2x^4}}$$

At stationary value V, let $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$

i.e. $$\frac{32x(9 - x^2)}{\sqrt{36x^2 - 2x^4}} = 0$$

$$32x(9 - x^2) = 0$$

$$x(3 - x)(3 + x) = 0$$

$$x = 0 \text{ or } x = \pm 3$$

From context, $x > 0$. Hence $x = 3$.

Substitute $x = 3$ into (4)

Stationary $V = 8\sqrt{36(3)^2 - 2(3^4)}$

$= 8\sqrt{162}$

$= 8\sqrt{81(2)}$

$= 8(9)\sqrt{2}$

$= 72\sqrt{2}$

$\therefore$ Stationary V is $72\sqrt{2}$ cm^3 (Shown)

Use First Derivative Test to show V is maximum

x	2.9	3	3.1
$\frac{dV}{dx}$	4.31	0	–4.77
Slope	/	—	\

Hence V is maximum when $x = 3$

(c) Let $h, 2r, W$ be the depth of candle, side length of candle surface, volume of candle remaining after burning respectively.

From **(1)** $l = \frac{36}{x} - x$

When $x = 3$ (given)

$l = \frac{36}{3} - 3$

$= 9$

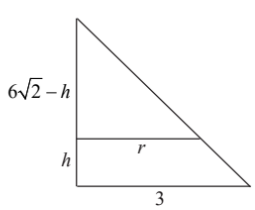
Substitute $l = 9$ and $x = 3$ into (2)

$H = \sqrt{9^2 - 3^2}$

$= 6\sqrt{2}$

Refer to the diagram.

By similar triangles,



$\frac{r}{6\sqrt{2} - h} = \frac{3}{6\sqrt{2}}$

$r = \frac{1}{2\sqrt{2}}(6\sqrt{2} - h)$

W = Volume of maximum candle $-$ Volume of candle burn

$$= 72\sqrt{2} - \frac{1}{3}(\text{Base area})(\text{Height})$$

$$= 72\sqrt{2} - \frac{1}{3}(2r)^2(6\sqrt{2} - h)$$

$$= 72\sqrt{2} - \frac{1}{3}\left(\frac{1}{2}\right)(6\sqrt{2} - h)^3$$

$$= 72\sqrt{2} - \frac{1}{6}(6\sqrt{2} - h)^3 \text{ .... (5)}$$

Differentiate (5) with respect to t

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{1}{6}(3)(6\sqrt{2} - h)^2(-1)\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{2}(6\sqrt{2} - h)^2\frac{\mathrm{d}h}{\mathrm{d}t}$$

Given that the candle burns at a rate of 10 cm^3 per second, i.e. $\frac{\mathrm{d}W}{\mathrm{d}t} = 10$

$$\therefore \quad 10 = \frac{1}{2}(6\sqrt{2} - h)^2\frac{\mathrm{d}h}{\mathrm{d}t} \text{ .... (6)}$$

Volume of candle remaining after burning after 6 seconds, W

$= 10 \text{ cm}^3$ per second $\times 6$

$= 60$

Substitute $w = 60$ into (5)

$$60 = 72\sqrt{2} - \frac{1}{6}(6\sqrt{2} - h)^3$$

$$h = 2.1778 \quad \text{(correct to 5 sf)}$$

$\therefore$ the height of the candle after burning after 6 seconds, $h = 2.1778$

Substitute $h = 2.1778$ into (6)

$$10 = \frac{1}{2}(6\sqrt{2} - 2.1778)^2\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.503 \text{ (correct to 3 sf)}$$

The rate of change of the height of the candle after 6 seconds is increasing at 0.503 cm per second

50
Solution

(a) Refer to the diagram.

Using Trigonometric ratio

$\cos\theta = \dfrac{AB}{HB}$ $\quad\quad$ $\sin\theta = \dfrac{BC}{BK}$

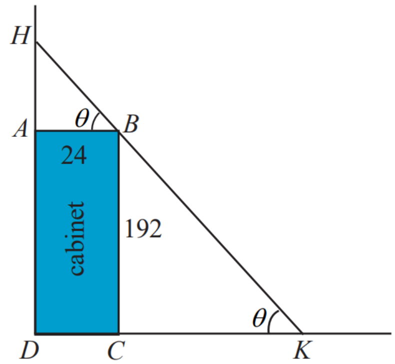
$\cos\theta = \dfrac{24}{HB}$ $\quad\quad$ $\sin\theta = \dfrac{192}{BK}$

$HB = \dfrac{24}{\cos\theta}$ $\quad\quad$ $BK = \dfrac{192}{\sin\theta}$

$\therefore\ L = HB + BK$

$= \dfrac{24}{\cos\theta} + \dfrac{192}{\sin\theta}$ (Shown)

$L = 24\sec\theta + 192\operatorname{cosec}\theta$ .... (1)



(b) Differentiate (1) with respect to θ

$$\frac{dL}{d\theta} = 24\sec\theta\tan\theta - 192\operatorname{cosec}\theta\cot\theta$$

When L is shortest, let $\dfrac{dL}{d\theta} = 0$

i.e. $24\sec\theta\tan\theta - 192\operatorname{cosec}\theta\cot\theta = 0$

$$\frac{24\sin\theta}{\cos^2\theta} - \frac{192(\cos\theta)}{\sin^2\theta} = 0$$

$$\frac{24\sin\theta}{\cos^2\theta} = \frac{192(\cos\theta)}{\sin^2\theta}$$

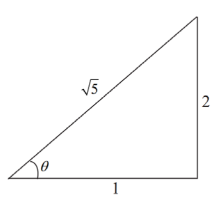
$$\tan^3\theta = 8$$

$$\tan\theta = 2 \text{ (since } \theta \text{ is acute)}$$

Refer to the triangle

i.e. $\operatorname{cosec}\theta = \dfrac{\sqrt{5}}{2}$

$\sec\theta = \sqrt{5}$



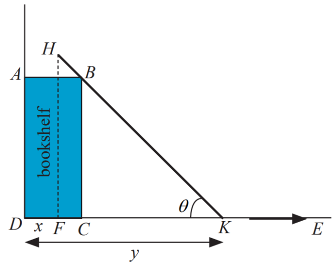
Substitute (2) into (1)

$$= 24\left(\frac{\sqrt{5}}{1}\right) + 192\left(\frac{\sqrt{5}}{2}\right)$$

$$= 120\sqrt{5}$$

Shortest length of the ladder is $120\sqrt{5}$ cm

(c) Refer to the diagram.



$DK - DF = FK$

$y - x = FK$ .... (1)

Using Trigonometric Ratio

$$\cos\theta = \frac{FK}{HK}$$

$$\cos\theta = \frac{FK}{L}$$

$$FK = L\cos\theta$$

Given $L = 270$,

$\therefore \quad FK = 270\cos\theta$ .... (2)

Equating (1) and (2)

$y - x = 270\cos\theta$ .... (3)

Differentiate (3) with respect to t

$$\frac{dy}{dt} - \frac{dx}{dt} = -270\sin\theta\frac{d\theta}{dt}$$

Since θ is acute, $\sin\theta > 0$ and $\dfrac{d\theta}{dt} < 0$

$\therefore \quad -270\sin\theta\dfrac{d\theta}{dt} > 0$

So, $\dfrac{dy}{dt} - \dfrac{dx}{dt} > 0$

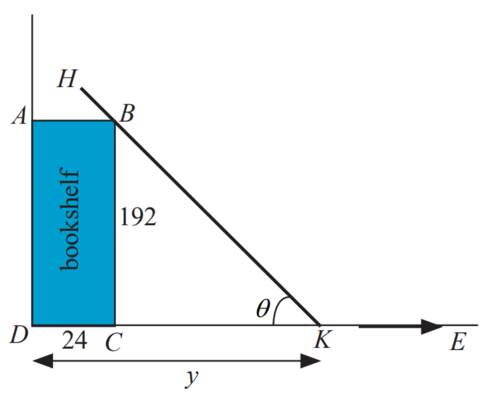
i.e. $\dfrac{dy}{dt} > \dfrac{dx}{dt}$

$\therefore$ the rate of change of y is greater than the rate of change of x.

(d) From (3): $y - x = 270\cos\theta$

$x = y - 270\cos\theta$ .... (4)

Refer to the diagram



$DK = DC + CK$

$y = 24 + \dfrac{192}{\tan\theta}$ .... (5)

Substitute (5) into (4)

$x = 24 + \dfrac{192}{\tan\theta} - 270\cos\theta$ .... (6)

Differentiate (6) with respect to θ

$$\frac{dx}{d\theta} = -\frac{192}{\tan^2\theta}\sec^2\theta + 270\sin\theta$$

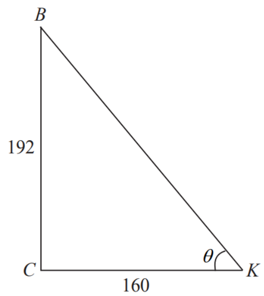
$$= -\frac{192}{\sin^2\theta} + 270\sin\theta$$

Given $CK = 160$,

$$\tan\theta = \frac{CB}{CK}$$

$$\tan\theta = \frac{192}{160} = \frac{6}{5}$$

$$\therefore \ \sin\theta = \frac{6}{\sqrt{61}}$$



Given that θ is decreasing at 0.1 radians per second, i.e. $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -0.1$

Using Chain Rule,

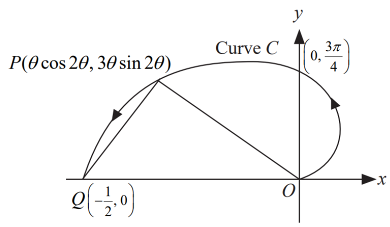
$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t} \\
&= \left(-\frac{192}{\sin^2\theta} + 270\sin\theta\right) \times (-0.1) \\
&= \left[270\left(\frac{6}{\sqrt{61}}\right) - \frac{192}{\left(\frac{6}{\sqrt{61}}\right)^2}\right] \times (-0.1) \\
&= 11.791
\end{aligned}$$

Rate of change of x is 11.8 cm s^{-1}.

51
Solution

(a) Point P has coordinates $(\theta\cos2\theta,\ 3\theta\sin2\theta)$.

Let area of triangle OPQ be A.



$$A = \frac{1}{2}\left(\frac{\pi}{2}\right)(3\theta\sin 2\theta)$$

$$= \frac{3\pi}{4}(\theta\sin 2\theta) \text{ .... (1)}$$

Differentiate (1) with respect to θ

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta) \text{ .... (2)}$$

Given that θ is increasing at the rate of 0.01 radians/sec, i.e. $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.1$

Using Chain Rule,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$= \frac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta)(0.01)$$

When $\theta = \frac{\pi}{6}$,

$$= \frac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta)(0.01)$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0.0327 \text{ units}^2 \quad \text{(correct to 3 s.f.)}$$

The rate of change of A increasing when $\theta = \frac{\pi}{6}$ is 0.0327 units^2/s

(b) For A to be stationary, let $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$

i..e. $\frac{3\pi}{4}(2\theta\cos 2\theta + \sin 2\theta) = 0$

$$2\theta\cos 2\theta + \sin 2\theta = 0$$

Using GC, $\theta = 1.0144$ or 1.01

Differentiate (2) with respect to θ

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\theta^2} = \frac{3\pi}{4}(-4\theta\sin 2\theta + 2\cos 2\theta + 2\cos 2\theta)$$

When $\theta = 1.0144$,

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\theta^2} = -12.7 < 0$$

Since $\frac{\mathrm{d}^2 A}{\mathrm{d}\theta^2} < 0$ when $\theta = 1.0144$, the value will result in maximum A.

Substitute $\theta = 1.0144$ into (1)

$$A = \frac{3\pi}{4}(1.0144)\sin(2\times 1.0144)$$
$$= 2.14 \text{ units}^2 \text{ (correct to 3 s.f.)}$$

$\therefore$ the value of A is 2.14 units^2

Substitute $\theta = 1.0144$ into $x = \theta\cos 2\theta$ and $y = 3\theta\sin 2\theta$

$\therefore$ $x = 1.0144\cos[2(1.0144)] = -0.449$

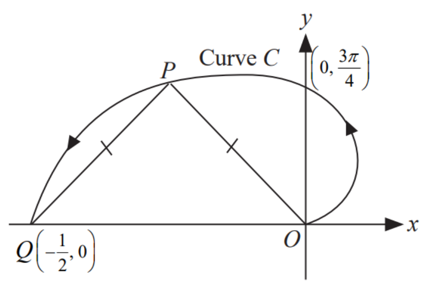
$y = 3(1.0144)\sin[2(1.0144)] = 2.73$

$\therefore$ the location of the drone is at a point with coordinates $(-0.449,\ 2.73)$

(c) For the drone is equidistant from points O and Q, then triangle OPQ is to be an isosceles triangle.

Mid-point of OQ, $x = -\frac{\pi}{2} \div 2$

$= -\frac{\pi}{4}$



Substitute $x = -\frac{\pi}{4}$ into $x = \theta\cos 2\theta$

$$-\frac{\pi}{4} = \theta\cos 2\theta$$

Using GC, $\theta = 1.1581$, where $0 \le \theta \le \frac{\pi}{2}$

Substitute $\theta = 1.1581$ into $y = 3\theta\sin 2\theta$

$y = 3(1.1581)\sin(2\times 1.1581)$

$= 2.55$

$\therefore$ the coordinates of the position where the drone is equidistant from points O and Q are $(-0.785,\ 2.55)$

52
Solution

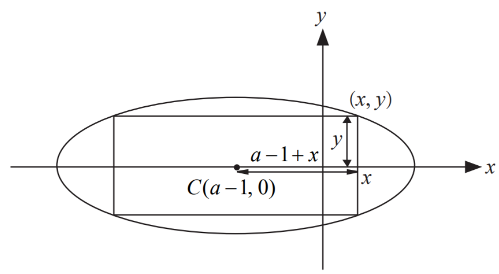
(a) Given $\dfrac{(x+a-1)^2}{a^2}+\dfrac{y^2}{b^2}=1$

Express y in terms of a, b and x

$$\frac{y^2}{b^2}=1-\frac{(x+a-1)^2}{a^2}$$

$$y^2=b^2-\frac{b^2(x+a-1)^2}{a^2}$$

$$y=\pm\sqrt{b^2-\frac{b^2}{a^2}(x+a-1)^2} \text{ .... (1)}$$



Let A be the area of rectangle.

$$A=2(x-1+a)2y \text{ .... (2)}$$

Substitute (1) into (2)

$$A=2(x-1+a)2\sqrt{b^2-\sqrt{b^2-\frac{b^2}{a^2}(x+a-1)^2}}$$

$$=4(x-1+a)\sqrt{b^2-\frac{b^2}{a^2}(x+a-1)^2}$$

$$=4(x-1+a)\sqrt{\frac{a^2b^2-b^2(x+a-1)^2}{a^2}} \quad \text{(Shown) .... (3)}$$

From (3): $A=4(x-1+a)\sqrt{b^2-\dfrac{b^2}{a^2}(x+a-1)^2}$

$$A^2=16(x-1+a)^2\left(b^2-\frac{b^2}{a^2}(x+a-1)^2\right) \text{ .... (4)}$$

Differentiate (4) with respect to x

$$2A\frac{\mathrm{d}A}{\mathrm{d}x}=32(x-1+a)\left(b^2-\frac{b^2}{a^2}(x-1+a)^2\right)-32(x-1+a)^2\frac{b^2}{a^2}(x-1+a)$$

$$2A\frac{\mathrm{d}A}{\mathrm{d}x}=32(x-1+a)\left[b^2-\frac{2b^2}{a^2}(x-1+a)^2\right]$$

At stationary point, $\dfrac{\mathrm{d}A}{\mathrm{d}x}=0$

$$2A(0)=32(x-1+a)\left[b^2-\frac{2b^2}{a^2}(x-1+a)^2\right]$$

$$0=32(x-1+a)b^2\left[1-\frac{2}{a^2}(x-1+a)^2\right]$$

$$0=(x-1+a)\left[1-\frac{2}{a^2}(x-1+a)^2\right]$$

$\therefore\ x=a-1$ (rejected as $x>1-a$) or $(x-1+a)^2=\dfrac{a^2}{2}$

$x=1-a+\dfrac{a}{\sqrt{2}}$ (rejected as $x>1-a$) or $x=1-a-\dfrac{a}{\sqrt{2}}$

Use Second Derivative Test to show A is maximum

Differentiate (4) with respect to x

$$2A\frac{\mathrm{d}A}{\mathrm{d}x} = 32(x-1+a)[b^2 - \frac{2b^2}{a^2}(x-1+a)^2]$$

$$2\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^2 + 2A\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 32[b^2 - \frac{2b^2}{a^2}(x-1+a)^2] + 32(x-1+a)(-4\frac{b^2}{a^2}(x-1+a))$$

$$2\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^2 + 2A\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 32[b^2 - \frac{2b^2}{a^2}(x-1+a)^2 - \frac{4b^2}{a^2}(x-1+a)^2]$$

$$2\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^2 + 2A\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 32[b^2 - \frac{6b^2}{a^2}(x-1+a)^2]$$

When $x = 1-a+\frac{a}{\sqrt{2}}$, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$, $A > 0$ and $b^2 - \frac{6b^2}{a^2}(x-1+a)^2 < 0$

$$\therefore \frac{\mathrm{d}^2A}{\mathrm{d}x^2} < 0$$

Area is maximum when $x = 1-a+\frac{a}{\sqrt{2}}$

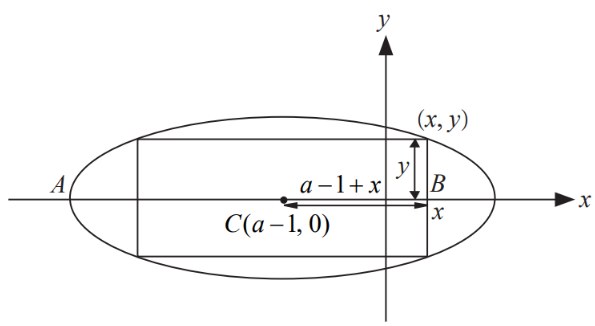
Horizontal width

$= AB$

$= \left(1-a+\frac{a}{\sqrt{2}}\right) - \left(1-a-\frac{a}{\sqrt{2}}\right)$

$= \frac{2a}{\sqrt{2}}$

$= \sqrt{2}a$ (Shown)



(b) Refer to the diagram.

Use Pythagoras Theorem,

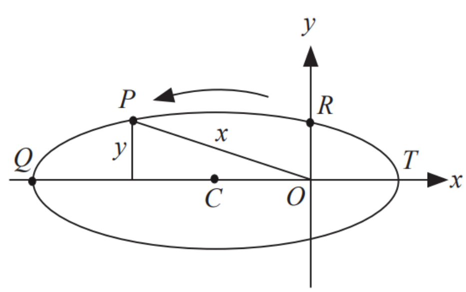
$OP^2 = x^2 + y^2$ .... (5)

Substitute (1) into (5)

$$OP^2 = x^2 + \left[b^2 - \frac{b^2}{a^2}(x-a-1)^2\right]$$

$$= \frac{1}{a^2}[a^2x^2 + a^2b^2 - b^2(x+a-1)^2]$$

$$OP = \frac{1}{a}\sqrt{a^2x^2 + a^2b^2 - b^2(x+a-1)^2} \quad \text{(Shown)}$$



(c) Let p denote the length OP.

$$p^2 = \frac{1}{a^2}[a^2x^2 + a^2b^2 - b^2(x+a-1)^2]$$

Differentiate (3) with respect to t

$$2p\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{a^2}\left[2a^2x\frac{\mathrm{d}x}{\mathrm{d}t} - 2b^2(x+a-1)\frac{\mathrm{d}x}{\mathrm{d}t}\right]$$

The x-component of the cyclist's velocity is equal to the camera's velocity, i.e. $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1-a}$

$$2p\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{a^2}\left[2a^2x \times \frac{1}{1-a} - 2b^2(x+a-1) \times \frac{1}{1-a}\right]$$

At $x = 1 - a + \frac{a}{\sqrt{2}}$ and when $a = 4$ and $b = 1$

$$2\left(\frac{1}{4}\right)\sqrt{4^2\left((1-4+\frac{4}{\sqrt{2}}\right)+4^2 1^2 - 1^2((1-4+\frac{4}{\sqrt{2}})+4-1)^2}\,\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{4^2}\left[2(4^2)\left(1-4+\frac{4}{\sqrt{2}}\right)\left(\frac{1}{1-4}\right) - 2(1^2)\left(1-4+\frac{4}{\sqrt{2}}+4-1\right)\left(\frac{1}{1-4}\right)\right]$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0.15958$$

$$= 0.160$$

$\therefore$ the rate of change of the distance OP when $x = 1 - a + \frac{a}{\sqrt{2}}$, $a = 4$ and $b = 1$ is 0.160 units per second.

53
Solution

(a)(i)

Let A cm^2 be the surface area of the cylindrical container.

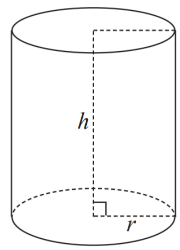
Let r cm and h cm be the radius and height of the cylindrical container respectively.

Volume $= \pi r^2 h$

Given that the volume of the cylindrical container is fixed at k cm^3,

i.e. $k = \pi r^2 h$

$\therefore \quad h = \dfrac{k}{\pi r^2}$ .... (1)



$A = 2\pi rh + \pi r^2$ .... (2)

Substitute (1) into (2)

$= 2\pi r\left(\dfrac{k}{\pi r^2}\right) + \pi r^2$

$= \dfrac{2k}{r} + \pi r^2$ .... (3)

Differentiate (3) with respect to r

$\dfrac{\mathrm{d}A}{\mathrm{d}r} = -\dfrac{2k}{r^2} + 2\pi r$

For the cylindrical container uses the least sheet metal, let $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 0$

i.e. $-\dfrac{2k}{r^2} + 2\pi r = 0$

$$2\pi r = \frac{2k}{r^2}$$

$$r^3 = \frac{k}{\pi}$$

$$r = \sqrt[3]{\frac{k}{\pi}}$$

Substitute $r = \sqrt[3]{\dfrac{k}{\pi}}$ into (1)

$\therefore \quad h = \dfrac{k}{\pi r^2}$

$$= \frac{k}{\pi\left[\left(\frac{k}{\pi}\right)^{\frac{1}{3}}\right]^2}$$

$$= \sqrt[3]{\frac{k}{\pi}}$$

Since $h = r,$ the height of cylindrical container is equal to its height. (Shown)

$\dfrac{\mathrm{d}^2 A}{\mathrm{d}r^2} = \dfrac{4k}{r^3} + 2\pi > 0$ since $p > 0$ and $k > 0$

Hence A is a minimum when $r = \sqrt[3]{\dfrac{k}{\pi}}$

(a)(ii)

From (2): $A = 2\pi rh + \pi r^2$

From **(a)**, $h : r = 1:1$, i.e. $h = r$

$$A = 2\pi r(r) + \pi r^2$$
$$= 3\pi r^2$$

For new design, $h : r = 5:2$, i.e. $h = \frac{5}{2}r$ .... (4)

Hence new $A = 2\pi rh + \pi r^2$

$$= 2\pi r\left(\frac{5}{2}r\right) + \pi r^2$$
$$= 6\pi r^2$$

The fraction of the area of the new design to the area of the old design $= \frac{6\pi r^2}{3\pi r^2} = \frac{2}{1}$

$\therefore$ the ratio of sheet metal used in this new design container to the sheet metal used in part **(a)(i)** is $2:1$

(b) Let V cm^3 be the volume of the plastic cylindrical container.

Method 1

$V = \pi r^2 h$ .... (5)

Substitute (4) into (5)

$$= \pi r^2\left(\frac{5}{2}r\right)$$
$$= \frac{5}{2}\pi r^3 \text{ .... (6)}$$

Let A cm^2 be the total surface area of the plastic container

$A = 2\pi rh + \pi r^2$ .... (7)

Substitute (4) into (6)

$$A = 2\pi r\left(\frac{5}{2}r\right) + \pi r^2$$
$$= 6\pi r^2 \text{ .... (8)}$$

Differentiate (6) with respect to r

$$\frac{dV}{dr} = \frac{15}{2}\pi r^2$$

Differentiate (8) with respect to r

$$\frac{dA}{dr} = 12\pi r$$

Using Chain Rule,

$$\frac{dA}{dr} = \frac{dA}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= 12\pi r \times \frac{2}{15\pi r^2} \times 80$$

$$= \frac{128}{r}$$

Given $h = 50$, substitute $h = 50$ into (4)

$$r = \frac{2}{5}(50)$$

$$= 20$$

When $r = 20$

Hence $\frac{dA}{dt} = \frac{128}{20} = 6.4 \text{ cm}^2/\text{s}$

Method 2

$A = 6\pi r^2 \therefore$

$$r = \sqrt{\frac{A}{6\pi}} \quad \text{or} \quad r = -\sqrt{\frac{A}{6\pi}} \quad \text{(rejected since } r \geq 0)$$

$$V = \pi r^2 h$$

$$= \pi r^2 \left(\frac{5}{2} r\right)$$

$$= \frac{5}{2}\pi r^3$$

$$= \frac{5}{2}\pi \left(\frac{A}{6\pi}\right)^3$$

$$= \frac{5A^{\frac{3}{2}}}{2(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$$

Differentiate V with respect to A

$$\frac{dV}{dA} = \frac{15A^{\frac{1}{2}}}{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$$

When $h = 50$,

$$r = \frac{2}{5}(50)$$

$$= 20$$

When $r = 20$

$\therefore \quad A = 6\pi(20)^2$

$$= 2400\pi$$

Using Chain Rule,

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$=\frac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15A^{\frac{1}{2}}}\times 80$$

When $A=2400\pi$

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15(2400\pi)^{\frac{1}{2}}}\times 80$$

$$=6.4\ \text{cm}^2/\text{s}$$

The rate of change of the surface area of the plastic container when its height is 50 cm is $6.4\ \text{cm}^2/\text{s}$.

Exercise 10

K Higher Order Questions

54
Solution

(a) Volume of the container = area of equilateral triangle $\times h$

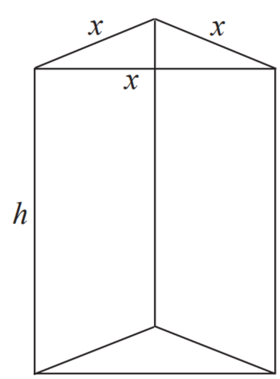
$$= \left(\frac{1}{2}x^2 \sin 60^\circ\right)h$$

$$= \frac{\sqrt{3}}{4}x^2 h$$

Gven that the volume of the container is k cm^3

$$\therefore \quad k = \frac{\sqrt{3}}{4}x^2 h$$

$$h = \frac{4k}{\sqrt{3}x^2} \quad \text{..............................} \ (1)$$



A = (area of 3 rectangular sides) + (area of 1 equilateral triangle)

$$A = 3(xh) + \frac{\sqrt{3}x^2}{4}$$

$$A = 3x\left(\frac{4k}{\sqrt{3}x^2}\right) + \frac{\sqrt{3}x^2}{4}$$

$$A = \frac{4k\sqrt{3}}{x} + \frac{\sqrt{3}x^2}{4} \quad \text{(Shown) ....} \ (2)$$

(b) Differentiate (2) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = -\frac{4\sqrt{3}k}{x^2} + \frac{\sqrt{3}x}{2} \quad \text{..............................} \ (3)$$

For minimum value of A, let $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$.

$$\therefore \quad -\frac{4\sqrt{3}k}{x^2} + \frac{\sqrt{3}x}{2} = 0$$

$$\frac{\sqrt{3}x}{2} = \frac{4\sqrt{3}k}{x^2}$$

$$x^3 = 8k$$

$$x = 2k^{\frac{1}{3}}$$

Substitute $x = 2k^{\frac{1}{3}}$ into (1)

$$h = \frac{4k}{\sqrt{3}\left(2k^{\frac{1}{3}}\right)^2}$$

$$= \frac{\sqrt[3]{k}}{\sqrt{3}}$$

Differentiate (3) with respect to x

$$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = \frac{8\sqrt{3}k}{x^3} + \frac{\sqrt{3}}{2}$$

For $k, x > 0$, $\dfrac{\mathrm{d}^2 A}{\mathrm{d}x^2} = \dfrac{8\sqrt{3}k}{x^3} + \dfrac{\sqrt{3}}{2} > 0$

Thus, A is a minimum.

55
Solution

Let the volume of the can be V.

$$V = \pi r^2 h$$

Given that closed cylindrical can that will hold 1500 cm^3 of liquid, i.e. $V = 1500$

$$\therefore \quad 1500 = \pi r^2 h$$

$$h = \frac{1500}{\pi r^2} \text{ .... (1)}$$

$$A = \text{curved surface area} + \text{area of 2 circular face}$$

$$= 2\pi r h + 2\pi r^2 \text{ .... (2)}$$

Substitute (1) into (2)

$$A = 2\pi r\left(\frac{1500}{\pi r^2}\right) + 2\pi r^2$$

$$= \frac{3000}{r} + 2\pi r^2$$

Differentiate (2) with respect to r

$$\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{3000}{r^2} + 4\pi r \text{ .... (3)}$$

When A is minimum, $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$.

$$\therefore \quad -\frac{3000}{r^2} + 4\pi r = 0$$

$$r^3 = \frac{3000}{4\pi}$$

$$= \frac{750}{\pi}$$

$$r = \sqrt[3]{\frac{750}{\pi}}$$

Use Second Derivative Test to show A is minimum

$$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = \frac{6000}{r^3} + 4\pi \text{ .... (3)}$$

When $r = \sqrt[3]{\frac{750}{\pi}}$,

$$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = \frac{6000}{\frac{750}{\pi}} + 4\pi$$

$$= 12\pi > 0$$

Hence, A is minimum when $r = \sqrt[3]{\frac{750}{\pi}}$.

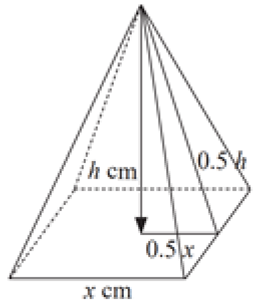
Substitute $r = \sqrt[3]{\frac{750}{\pi}}$ into (1)

$$h = \frac{1500}{\pi\left(\sqrt[3]{\frac{750}{\pi}}\right)^2}$$

$$= 12.4 \quad \text{(correct to 3 s.f.)}$$

$\therefore\ r = \sqrt[3]{\frac{750}{\pi}} =$ and $h = 12.4$

56
Solution



Let the volume of the pyramid by V.
By Pythagoras' Theorem,

$$h^2+\left(\frac{1}{2}x\right)^2=\left(\frac{1}{2}d\right)^2$$
$$h^2+\frac{x^2}{4}=\frac{d^2}{4}$$
$$x^2=d^2-4h^2 \quad \text{.............................. (1)}$$

Let the volume of pyramid be V

$$V=\frac{1}{3}\times(\text{Base area})\times(\text{vertical height})$$
$$=\frac{1}{3}x^2h \quad \text{.............................. (2)}$$

Substitute (1) into (2)

$$=\frac{1}{3}(d^2-4h^2)h$$
$$=\frac{1}{3}d^2h-\frac{4}{3}h^3 \quad \text{.............................. (3)}$$

Differentiate (3) with respect to h

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}d^2-4h^2$$

For maximum volume of the pyramid, let $\frac{\mathrm{d}V}{\mathrm{d}h}=0$

i.e. $\frac{1}{3}d^2-4h^2=0$

$$\frac{1}{3}d^2=4h^2$$
$$h^2=\frac{d^2}{12}$$
$$h=\sqrt{\frac{d^2}{12}} \quad (\text{since } h>0)$$
$$=\frac{d}{2\sqrt{3}}$$

Substitute $h=\frac{d}{2\sqrt{3}}$ into (3)

$$V=\frac{1}{3}h(d^2-4h^2)$$
$$=\frac{1}{3}\left(\frac{d}{2\sqrt{3}}\right)\left(d^2-\frac{4d^2}{12}\right)$$
$$=\frac{d^3}{9\sqrt{3}}$$

$\therefore$ the maximum volume of the pyramid is $\frac{d^3}{9\sqrt{3}}$

57
Solution

Volume of the model

$= (\text{Volume of the cylinder}) + (\text{Volume of the cone})$

$= \pi(15x)^2 w + \frac{1}{3}\pi(15x)^2(8x)$

Given that the volume of the model is $k\ \text{cm}^3$

$$\therefore \quad k = \pi(15x)^2 w + \frac{1}{3}\pi(15x)^2(8x)$$

$$w = \frac{k - 600\pi x^3}{225\pi x^2} \ \text{.............................. (1)}$$

Let the external surface area be A

$A = \text{curved surface area of the cylinder} + \text{circular base area} + \text{curved surface area of the cone}$

$$A = 2\pi(15x)w + \pi(15x)^2 + \pi(15x)\sqrt{(8x)^2 + (15x)^2}$$

$$= 2\pi 15x)w + \pi(15x)^2 + \pi(15x)(17x)$$

$$= 30\pi xw + 480\pi x^2 \ \text{.............................. (2)}$$

Substitute (1) into (2)

$$A = 30\pi x\left[\frac{k - 600\pi x^3}{225\pi x^2}\right] + 480\pi x^2$$

$$= \frac{2k}{15x} + 400\pi x^2 \ \text{.............................. (3)}$$

Differentiate (3) with respect to x

$$\frac{\mathrm{d}A}{\mathrm{d}x} = -\frac{2k}{15x^2} + 800\pi x$$

For minimum external surface area, let $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$.

i.e. $-\frac{2k}{15x^2} + 800\pi x = 0$

$$\frac{2k}{15x^2} = 800\pi x$$

$$x^3 = \frac{k}{6000\pi}$$

$$x = \frac{1}{10}\left(\frac{k}{6\pi}\right)^{\frac{1}{3}}$$

Substitute $x = \frac{1}{10}\left(\frac{k}{6\pi}\right)^{\frac{1}{3}}$ into (1)

$$w = \frac{k - 600\pi\left[\frac{1}{10}\left(\frac{k}{6\pi}\right)^{\frac{1}{3}}\right]^3}{225\pi\left(\frac{1}{10}\left(\frac{k}{6\pi}\right)^{\frac{1}{3}}\right)^2}$$

$$w=\frac{k-600\pi\times\left(\frac{k}{6000\pi}\right)}{\frac{225\pi k^{\frac{2}{3}}}{100(6^{\frac{2}{3}})(\pi^{\frac{2}{3}})}}$$

$$=\frac{\frac{9k}{10}}{\frac{225\pi k^{\frac{2}{3}}}{100(6^{\frac{2}{3}})(\pi^{\frac{2}{3}})}}$$

$$=\frac{9k}{10}\times\frac{100\left(6^{\frac{2}{3}}\right)\left(\pi^{\frac{2}{3}}\right)}{225\pi k^{\frac{2}{3}}}$$

$$=\frac{2k^{\frac{1}{3}}}{5}\times\frac{6\times 6^{\frac{1}{3}}\left(\pi^{\frac{2}{3}}\right)}{\pi}$$

$$=\frac{12}{5}\left(\frac{k}{6\pi}\right)^{\frac{1}{3}}$$

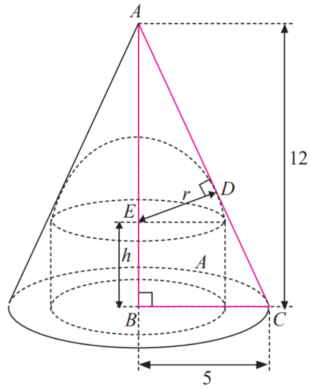
$$\therefore\ x=\frac{1}{10}\left(\frac{k}{6\pi}\right)^{\frac{1}{3}}\ \text{and}\ w=\frac{12}{5}\left(\frac{k}{6\pi}\right)^{\frac{1}{3}}$$

、

58
Solution

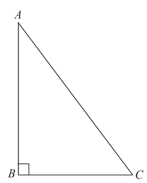
(a) Use Pythagoras Theorem,

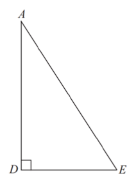
$$AB^2 + BC^2 = AC^2$$
$$12^2 + 5^2 = AC^2$$
$$\therefore \quad AC = \sqrt{5^2 + 12^2}$$
$$= 13$$



Using the diagram provided to identify that ΔABC is similar to ΔADE.
Using similar triangles

$$\therefore \quad \frac{AE}{DE} = \frac{AC}{BC}$$
$$\frac{AE}{r} = \frac{13}{5}$$
$$AE = \frac{13}{5}r$$





$$BE = AB - AE$$
$$h = AB - AE$$
$$= 12 - \frac{13}{5}r \quad \text{(Shown) .... (1)}$$

Since r and h are lengths, i.e. $r \geq 0$ and $h \geq 0$

From (1): $\quad 12 - \frac{13}{5}r \geq 0$

$$r \leq \frac{60}{13}$$

$$\therefore \quad 0 \leq r \leq \frac{60}{13}$$

(b) Volume of inscribed container,

V = Volume of the cylinder + Volume of the hemisphere

$$V = \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) \text{ .... (2)}$$

Substitute (1) into (2)

$$= \pi r^2\left(12 - \frac{13}{5}r\right) + \frac{2}{3}\pi r^3$$
$$= 12\pi r^2 - \frac{29}{15}\pi r^3 \text{ .... (3)}$$

Differentiate (3) with respect to r

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 24\pi r - \frac{29}{5}\pi r^2.$$

For maximum volume V, let $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$.

i.e. $24\pi r - \frac{29}{5}\pi r^2 = 0$

$\pi r\left(24 - \frac{29}{5}r\right) = 0$

$r = 0$ (rejected as $r \neq 0$) or $r = \frac{120}{29}$

Use First Derivative Test to show V is maximum

r	$r = \frac{120^-}{29}$ e.g. $r = \frac{119}{29}$	$r = \frac{120}{29}$	$r = \frac{120^+}{29}$ e.g. $r = \frac{121}{29}$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	$\frac{119}{145}\pi > 0$	0	$-\frac{121}{145}\pi < 0$

$\therefore$ maximum volume at $r = \frac{120}{29}$

59
Solution

(a) Observe that $PQ = QC = x$ and $PR = RC = y$.

Use Pythagoras Theorem,

$$PB^2 + BQ^2 = PQ^2$$

$$\begin{aligned} PB &= \sqrt{PQ^2 - BQ^2} \\ &= \sqrt{x^2 - (6-x)^2} \\ &= \sqrt{x^2 - 36 + 12x - x^2} \\ &= \sqrt{12x - 36} \\ &= 2\sqrt{3}\sqrt{x-3} \end{aligned}$$

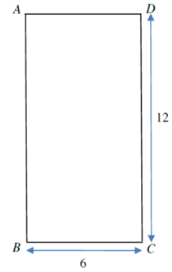


Fig. 1

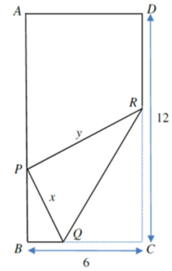


Fig. 2

(b) **Method 1**

Let R' be the foot of perpendicular from R to AB.

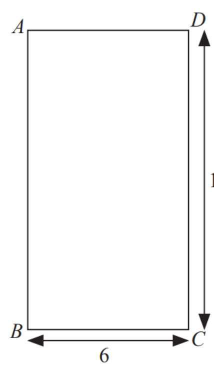
Since $\Delta RR'P$ and ΔPBQ are similar triangles,

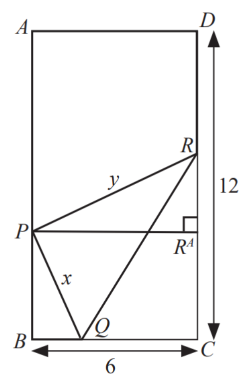
$$\frac{RP}{PQ} = \frac{RR'}{PB}$$

$$\frac{y}{x} = \frac{6}{\sqrt{12x-36}}$$

$$y = \frac{6x}{\sqrt{12(x-3)}}$$

$$\begin{aligned} y &= \frac{6x}{2\sqrt{3}\sqrt{x-3}} \\ &= \frac{3x}{\sqrt{3}\sqrt{x-3}} \\ &= x\sqrt{\left(\frac{3}{x-3}\right)} \quad \text{(Shown)} \end{aligned}$$





Method 2

Let R' be the foot of perpendicular from P to CD.

Then $\begin{aligned} RR' &= RC - P'C \\ &= y - PB \\ &= y - \sqrt{12x-36} \end{aligned}$

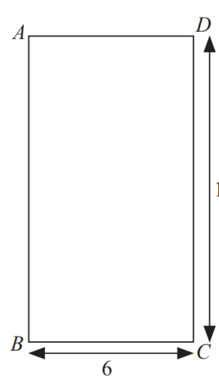
Consider $\Delta RPP'$, by Pythagoras Theorem,

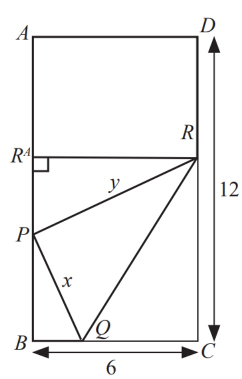
$$y^2 = 6^2 + (y - \sqrt{12x-36})^2$$

$$y^2 = 36 + y^2 - 2y\sqrt{12x-36} + (12x-36)$$

$$2y\sqrt{12x-36} = 12x$$

$$\begin{aligned} y &= \frac{6x}{\sqrt{12x-36}} \\ &= \frac{6x}{2\sqrt{3}\sqrt{x-3}} \\ &= \frac{3x}{\sqrt{3\sqrt{x-3}}} \\ &= x\sqrt{\left(\frac{3}{x-3}\right)} \quad \text{(Shown) .... (1)} \end{aligned}$$





(c) Let QR be l.

by Pythagoras Theorem,

$l^2 = x^2 + y^2$ .... (2)

Substitute (1) into (2)

$l^2 = x^2 + \dfrac{3x^2}{x-3}$ .... (3)

Differentiate (3) with respect to θ

$$2l\frac{\mathrm{d}l}{\mathrm{d}x} = 2x + \frac{(x-3)(6x)-3x^2}{(x-3)^2}$$

$$2l\frac{\mathrm{d}l}{\mathrm{d}x} = \frac{(2x)(x-3)^2+(x-3)(6x)-3x^2}{(x-3)^2}$$

$$2l\frac{\mathrm{d}l}{\mathrm{d}x} = \frac{2x(x^2-6x+9)+6x^2-18x-3x^2}{(x-3)^2}$$

$$2l\frac{\mathrm{d}l}{\mathrm{d}x} = \frac{2x^3-9x^2}{(x-3)^2} = \frac{x^2(2x-9)}{(x-3)^2}$$

For stationary value, when $\dfrac{\mathrm{d}l}{\mathrm{d}x} = 0$,

$\therefore \quad x^2(2x-9) = 0$

$x = 0$ (rejected since $x > 0$) or $x = \dfrac{9}{2}$

Use First Derivative Test to show I is minimum

When $x = \dfrac{9^-}{2}$, $2x-9<0$, $\dfrac{\mathrm{d}l}{\mathrm{d}x} = \dfrac{x^2(2x-9)}{2l(x-3)^2} < 0$

When $x = \dfrac{9^+}{2}$, $2x-9>0$, $\dfrac{\mathrm{d}l}{\mathrm{d}x} = \dfrac{x^2(2x-9)}{2l(x-3)^2} < 0$

Hence when $x = \dfrac{9}{2}$, l is minimum. (Proven)

Substitute $x = \dfrac{9}{2}$ into (3)

$$l^2 = x^2 + \frac{3x^2}{x-3}$$

$$= \frac{81}{4} + \frac{(3)\left(\frac{81}{4}\right)}{\frac{9}{2}-3}$$

$$= \frac{243}{4}$$

$$l = \sqrt{\frac{243}{4}}$$

$\therefore$ the minimum length of QR is $\dfrac{9\sqrt{3}}{2}$.

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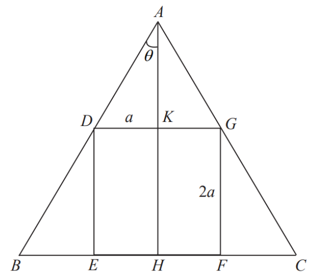
Solution

(a) Refer ΔAKD in the diagram.

Let the the height of triangle be AK.



$$\tan\theta = \frac{DK}{AK}$$

Taking $t = \tan\theta$,

$$t = \frac{a}{AK}$$

$$AK = \frac{a}{t}$$

Hence $AH = AK + KH$

$$= 2a + \frac{a}{t}$$

$$= a\left(2 + \frac{1}{t}\right)$$

Refer ΔDEB in the diagram.

$$\tan\theta = \frac{BE}{DE}$$

Taking $t = \tan\theta$,

$$t = \frac{BE}{2a}$$

$$BE = \frac{2a}{t}$$

Hence $BH = BE + EH$

$$= 2a\tan\theta + a$$

$$= a(2t+1)$$

Area $S = \frac{1}{2}(AH)(BC)$

$$S = \frac{a}{2}\left(2 + \frac{1}{t}\right)(2a(2t+1))$$

$$S = a^2\left(2 + \frac{1}{t}\right)(2t+1)$$

$$S = a^2\left(4 + 4t + \frac{1}{t}\right) \text{ .... (1) (Shown)}$$

(b) Differentiate (1) with respect to t

$$\frac{dS}{dt} = a^2\left(4 - \frac{1}{t^2}\right) \text{ .... (2)}$$

For the minimum area, let $\dfrac{dS}{dt} = 0,$

$$a^2\left(4 - \frac{1}{t^2}\right) = 0$$

$$t^2 = \frac{1}{4}$$

$$t = \frac{1}{2} \quad \text{or} \quad t = -\frac{1}{2} \quad \text{(Rejected, since } \theta \text{ is acute)}$$

Use Second Derivative Test to show S is minimum

Differentiate (2) with respect to t

$$\frac{d^2S}{dt^2} = a\left(\frac{2}{t^3}\right)$$

When $t = \dfrac{1}{2}$, $\dfrac{d^2S}{dt^2} = a^3\left(\dfrac{2}{\left(\frac{1}{2}\right)^3}\right) = 16a > 0.$

Hence the minimum value of S occurs when $t = \dfrac{1}{2}$.

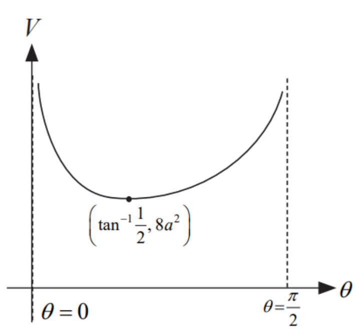
Substitute $t = \dfrac{1}{2}$ into (1)

$$S = a^2\left(4 + 4 \times \frac{1}{2} + 2\right)$$

$$= 8a^2$$

$\therefore$ the minimum S is $8a^2$.

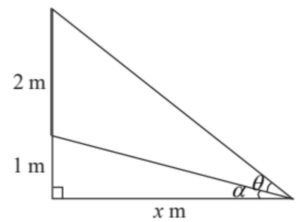
(c) The graph of $S = a^2\left(4 + 4\tan\theta + \dfrac{1}{\tan\theta}\right)$



61
Solution

Let α be defined as shown.

(a) Using Trigonometric Ratio,



$$\tan\alpha = \frac{1}{x}$$

$$\alpha = \tan^{-1}\frac{1}{x} \text{ ....}$$

Also $\tan(\theta+\alpha) = \frac{3}{x}$

$$\theta+\alpha = \tan^{-1}\frac{3}{x}$$

$$\therefore \quad \theta = \tan^{-1}\frac{3}{x} - \alpha \text{ .... (2)}$$

Substitute (1) into (2)

$$\theta = \tan^{-1}\frac{3}{x} - \tan^{-1}\frac{1}{x} \quad \text{(Shown) .... (3)}$$

(b) Differentiate (3) with respect to x

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{3}{x}\right)^2}\cdot\frac{-3}{x^2} - \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{-1}{x^2}$$

$$= -\frac{x^2}{x^2+3^2}\cdot\frac{3}{x^2} + \frac{x^2}{x^2+1}\cdot\frac{1}{x^2}$$

$$= \frac{1}{x^2+1} - \frac{3}{x^2+9}$$

$$= \frac{x^2+9-3x^2-3}{(x^2+1)(x^2+9)}$$

$$= \frac{6-2x^2}{(x^2+1)(x^2+9)} \quad \text{(Shown)}$$

$a = 6$ and $b = -2$

For stationary, let $\frac{d\theta}{dx} = 0$

i.e. $\frac{6-2x^2}{(x^2+1)(x^2+9)} = 0$

$$6-2x^2 = 0$$

$$\therefore \quad x = \sqrt{3} \quad \text{or} \quad x = -\sqrt{3} \quad (\text{rejected as } x > 0)$$

Use First Derivative Test to show θ is maximum

x	$\left(\sqrt{3}\right)^-$	$\sqrt{3}$	$\left(\sqrt{3}\right)^+$
$\frac{d\theta}{dx}$	$+$ ve	0	$-$ ve
	/	—	\

$\therefore \theta$ is maximum at $x = \sqrt{3}$ m.

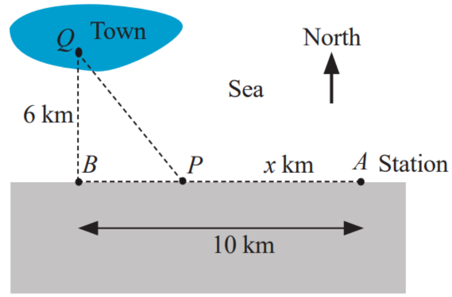
It is not realist as it is assumed that the observer's eye level is at ground level. (or The observer is viewing from the floor.)

62
Solution

(a) Distance BP

$= AB - BP$

$= 10 - x$ km



Use Pythagoras Theorem,

$PQ^2 = BQ^2 + BP^2$

$= (6^2 + (10-x)^2)$

$= \sqrt{(36 + 100 - 20x + x^2)}$

$\therefore$ Distance $PQ = \sqrt{(x^2 - 20x + 136)}$

C = Cost of underground cable + Cost of undersea cable

$= 125x + 1.6 \times 125\sqrt{(x^2 - 20x + 136)}$

$= 125x + 200\sqrt{(x^2 - 20x + 136)}$ .... (1) (Shown)

$\therefore\ k = 200$

(b) Differentiate (1) with respect to x

$$\frac{\mathrm{d}C}{\mathrm{d}x} = 125 + 200\left[\frac{1}{2}(x^2 - 20x + 136)^{-\frac{1}{2}}\right](2x - 20)$$

$$= 125 - \frac{200(10-x)}{(x^2 - 20x + 136)^{\frac{1}{2}}}$$

For statonary value, $\frac{\mathrm{d}C}{\mathrm{d}x} = 0$

$$125 - \frac{200(10-x)}{(x^2 - 20x + 136)^{\frac{1}{2}}} = 0$$

$$125(x^2 - 20x + 136)^{\frac{1}{2}} = 200(10 - x)$$

$$x^2 - 20x + 136 = \frac{64}{25}(10-x)^2$$

$$25(x^2 - 20x + 136) = 64(100 - 20x + x^2)$$

$$39x^2 - 780x + 3000 = 0$$

$$x = 10 - 10\sqrt{\frac{3}{13}} \quad \text{or} \quad x = 10 + 10\sqrt{\frac{3}{13}} \quad \text{(rejected as } 0 \le x \le 10)$$

Use First Derivative Test to verify that C is minimum

x	$\left(10-10\sqrt{\frac{3}{13}}\right)^-$	$10-10\sqrt{\frac{3}{13}}$	$\left(10-10\sqrt{\frac{3}{13}}\right)^+$
$\frac{\mathrm{d}C}{\mathrm{d}x}$	$-$ ve	0	$+$ ve
Slope	\	—	/

$\therefore$ C is minimum at $x = 10 - 10\sqrt{\frac{3}{13}}$ km.

Alteratively Method: (Use Second Derivative Test to verify that C is minimum)

$$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = 200\left[\frac{(x^2-20x+136)^{\frac{1}{2}}(-1)-(10-x)\frac{1}{2}(x^2-20x+136)^{-\frac{1}{2}}(2x-20)}{x^2-20x+136}\right]$$

$$= \frac{-200(x^2-20x+136)^{-\frac{1}{2}}(-(x^2-20x+136)+(10-x)^2)}{x^2-20x+136}$$

$$= \frac{-200(-x^2+20x-136+100-20x+x^2)}{(x^2-20x+136)^{\frac{3}{2}}}$$

$$= \frac{7200}{(x^2-20x+136)^{\frac{3}{2}}}$$

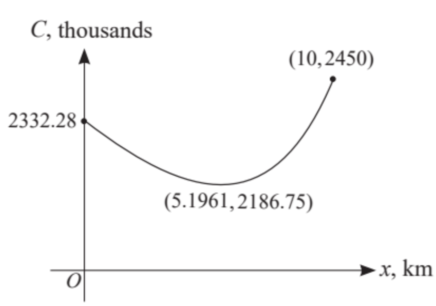
$$= \frac{7200}{(36+(10-x)^2)^{\frac{3}{2}}}$$

When $x = 10-10\sqrt{\frac{3}{13}}$, $(36+(10-x)^2) > 0$

$\therefore \quad \frac{\mathrm{d}^2C}{\mathrm{d}x^2} = \frac{7200}{(36+(10-x)^2)^{\frac{3}{2}}} > 0$

$\therefore$ C is minimum at $x = 10 - 10\sqrt{\frac{3}{13}}$ km.

(c)

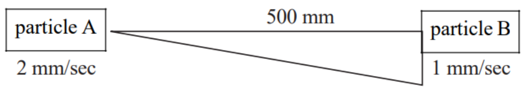


(d) Refer to the graph in **(c)**.

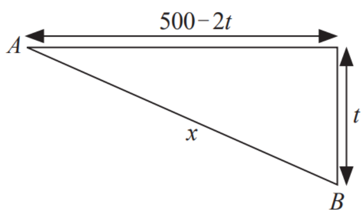
Max cost occurs at 2450 thousands.

63
Solution

(a)



The diagram above shows a diagrammatic representation of particle A and particle B at the beginning of an experiment.



The diagram above shows a diagrammatic representation of particle A and particle B after t seconds.

Use Pythagoras Theorem,

$$x = \sqrt{(500-2t)^2 + t^2}$$

Differentiate x with respect to t

$$\frac{dx}{dt} = \frac{1}{2}((500-2t)^2 + t^2)^{-\frac{1}{2}}(2(500-2t)(-2) + 2t)$$

$$= \frac{t - 1000 + 4t}{\sqrt{(500-2t)^2 + t^2}}$$

At $t = 45$ seconds (0.75 minutes)

$$\frac{dx}{dt} = \frac{5(45) - 1000}{\sqrt{(500-2(45))^2 + (45)^2}}$$

$$= -1.88$$

$\therefore$ the rate at which x decreasing at -1.88 mm per second after 0.75 minutes from the start of the experiment.

(b) From **(a)**: $\dfrac{dx}{dt} = \dfrac{t - 1000 + 4t}{\sqrt{(500-2t)^2 + t^2}}$

When x is at minimum, $\dfrac{dx}{dt} = 0$

$$\frac{5t - 1000}{\sqrt{(500-2t)^2 + t^2}} = 0$$

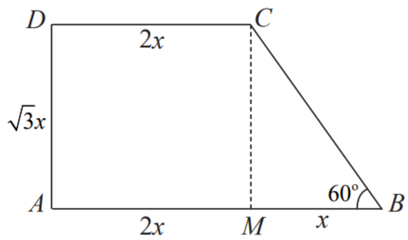
$$t = 200$$

$\therefore$ the value of t corresponding to the minimum value of x is 200.

64
Solution

(a) Refer to the diagram.

Using Trigonometric ratio,



$$\tan 60^\circ = \frac{CM}{BM}$$

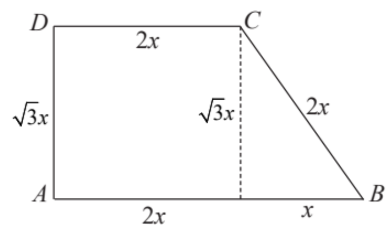
$$\sqrt{3} = \frac{CM}{x}$$

$$CM = \sqrt{3}x$$

$$\therefore \quad CM = AD = \sqrt{3}x$$

$$\cos 60^\circ = \frac{x}{BC}$$

$$BC = \frac{x}{\cos 60^\circ} = 2x$$



Let V be the volume of the trough.

$$V = \frac{1}{2}(2x+3x)\times\sqrt{3}x\times y = \frac{5\sqrt{3}x^2 y}{2}$$

Given that when the trough is filled to its maximum volume is $5\sqrt{3}k$ m^3, i.e. $V = 5\sqrt{3}k$

$$5\sqrt{3}k = \frac{5\sqrt{3}x^2}{2}y$$

$$y = \frac{2k}{x^2} \quad \text{.............................. (1)}$$

$$A = \frac{1}{2}(2x+3x)\times\sqrt{3}x\times 2 + 3xy + 2xy + \sqrt{3}xy \quad \text{.............................. (2)}$$

$$= 5\sqrt{3}x^2 + (5+\sqrt{3})xy$$

Substitute (1) into (2)

$$= 5\sqrt{3}x^2 + (5+\sqrt{3})\times\left(\frac{2k}{x^2}\right)$$

$$= 5\sqrt{3}x^2 + \frac{2(5+\sqrt{3})k}{x} \quad \text{(Shown) .... (3)}$$

Differentiate (3) with respect to x

$$\frac{dA}{dx} = 10\sqrt{3}x - \frac{2(5+\sqrt{3})k}{x^2}$$

For stationary A, let $\dfrac{\mathrm{d}A}{\mathrm{d}x}=0$

i.e. $10\sqrt{3}x-\dfrac{2(5+\sqrt{3})k}{x^2}=0$

$$5\sqrt{3}x=\frac{(5+\sqrt{3})k}{x^2}$$

$$x^3=\frac{(5+\sqrt{3})k}{5\sqrt{3}}$$

$$x=\left(\left(\frac{\sqrt{3}}{3}+\frac{1}{5}\right)k\right)^{\frac{1}{3}}$$

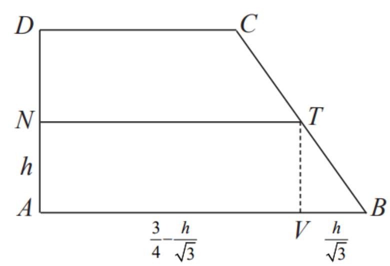
(b) Substitute $k=\dfrac{3}{160}$ and $y=0.6$ into (1)

$$0.6=\frac{2}{x^2}\times\frac{3}{160}$$

$$x=0.25$$

Hence, $AB=0.25\times 3=0.75=\dfrac{3}{4}$ and $AD=\sqrt{3}\times 0.25=\dfrac{\sqrt{3}}{4}$

Let V_o denotes the volume of water in the trough.



$$\begin{aligned}V_o&=(\text{Area of trapezium})\times\text{length}\\&=\frac{1}{2}\left(AB+NT\right)\times AN\times 0.6\\&=\frac{1}{2}\left(\frac{3}{4}+\frac{3}{4}-\frac{h}{\sqrt{3}}\right)\times h\times 0.6\\&=0.3h\left(\frac{3}{2}-\frac{h}{\sqrt{3}}\right)\\&=0.45h-\frac{0.3}{\sqrt{3}}h^2\ \text{.............................}\ (4)\end{aligned}$$

Given that when the trough is filled to its maximum volume is $5\sqrt{3}k\ \mathrm{m}^3$,

i.e. $V_o=5\sqrt{3}k$

Taking $k=\dfrac{3}{160}$, $\therefore\ V_o=5\sqrt{3}\times\dfrac{3}{160}$

When the volume of water in the trough is half filled, $V_o=\dfrac{1}{2}\left(5\sqrt{3}\times\dfrac{3}{160}\right)$. .... (5)

Equating (4) and (5)

$$\frac{1}{2}\left(5\sqrt{3}\times\frac{3}{160}\right)=\frac{3\sqrt{3}}{64}$$

$$0.45h-\frac{0.3}{\sqrt{3}}h^2=\frac{3\sqrt{3}}{64}$$

$$\frac{0.3}{\sqrt{3}}h^2-0.45h+\frac{3\sqrt{3}}{64}=0$$

Using GC, $h=0.19507$ or $h=2.40301$ (rejected since $h<AD\approx 0.433$)

Differentiate (4) with respect to x

$$\frac{\mathrm{d}V}{\mathrm{d}h} = 0.45 - \frac{0.6}{\sqrt{3}}h$$

Using Chain Rule,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$$

Water is pouring at a constant rate of $0.015\ \mathrm{m}^3$ per minute into the trough, i.e. $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.015$

$$0.015 = \left(0.45 - \frac{0.6}{\sqrt{3}} \times 0.19507\right) \times \frac{\mathrm{d}h}{\mathrm{d}t}$$

When $h = 0.19507$

$$0.015 = \left(0.45 - \frac{0.6}{\sqrt{3}} \times 0.19507\right) \times \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.03922$$

Hence the rate of change of height is 0.03922 m per minute.

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Solution

(a) Given $x = (v\cos\theta)t$ .... (1)

and $y = (v\sin\theta)t - 5t^2$ .... (2)

Differentiate (1) with respect to t

$$\frac{dx}{dt} = v\cos\theta$$

Differentiate (2) with respect to t

$$\frac{dy}{dt} = v\sin\theta - 10t$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{v\sin\theta - 10t}{v\cos\theta}$$

For greatest height, $\frac{dv}{dx} = 0.$

i.e. $\frac{v\sin\theta - 10t}{v\cos\theta} = 0$

$$t = \frac{v\sin\theta}{10}$$

Substitute $t = \frac{v\sin\theta}{10}$ into (2)

$$y = (v\sin\theta)\left(\frac{v\sin\theta}{10}\right) - 5\left(\frac{v\sin\theta}{10}\right)^2$$

$$= \frac{v^2\sin^2\theta}{20}$$

Hence, the greatest height at which the particle is at the highest point in the trajectory is $\frac{v^2\sin^2\theta}{20}$ metres

Alternative Method

For maximum height, we have zero vertical velocity, i.e. $\frac{dy}{dt} = 0$

$$v\sin\theta - 10t = 0$$

$$t = \frac{v\sin\theta}{10}$$

Substitute $t = \frac{v\sin\theta}{10}$ into (2)

$$y = (v\sin\theta)\left(\frac{v\sin\theta}{10}\right) - 5\left(\frac{v\sin\theta}{10}\right)^2$$

$$= \frac{v^2\sin^2\theta}{20} \text{ metres}$$

Hence, the greatest height at which the particle is at the highest point in the trajectory is $\frac{v^2\sin^2\theta}{20}$ metres

(b) Maximum height of particle from the ground $= \dfrac{v^2 \sin^2 \theta}{20} + 29$

Given that the greatest height that the particle reaches is 57.8 m.

$$\therefore \quad 57.8 = \frac{v^2 \sin^2 \theta}{20} + 29$$

$$576 = v^2 \sin^2 \theta$$

Since $v > 0$ and $0 < \theta < \dfrac{\pi}{2}$.

$$v \sin \theta = 24 \text{ .... } (3)$$

When the particle hits ground, $y = -29$.
Substitute $y = -29$ into (2)

$$-29 = (v \sin \theta)t - 5t^2$$

$$5t^2 - 24t - 29 = 0$$

$$(5t - 29)(t + 1) = 0$$

$$t = 5.8 \quad \text{or} \quad t = -1 \text{ (reject, } t > 0)$$

Hence, time taken for particle to hit the ground is 5.8 seconds.

When the particle hits the ground at A which is at a horizontal distance of 104.4 m from O, i.e. $x = 104.4$
Substitute $x = 104.4$ into (1)

$$104.4 = (v \cos \theta)t$$

$$v \cos \theta = \frac{104.4}{5.8}$$

$$v \cos \theta = 18 \text{ .... } (4)$$

Taking $(3)^2 + (4)^2$

$$v^2 \sin^2 \theta + v^2 \cos^2 \theta = 24^2 + 18^2$$

$$v^2 = 900$$

$$v = 30 \text{ since } v > 0.$$

The value of v when $t = 5.8$ is 30 ms^{-1}.

(c) When particle hits ground, $v \sin \theta = 24$, $v \cos \theta = 18$, $t = 5.8$.

Hence, $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v \sin \theta - 10t}{v \cos \theta}$$

$$= \frac{24 - 10(5.8)}{18}$$

$$= -\frac{17}{9}$$

The exact gradient of the tangent at A is $-\dfrac{17}{9}$.